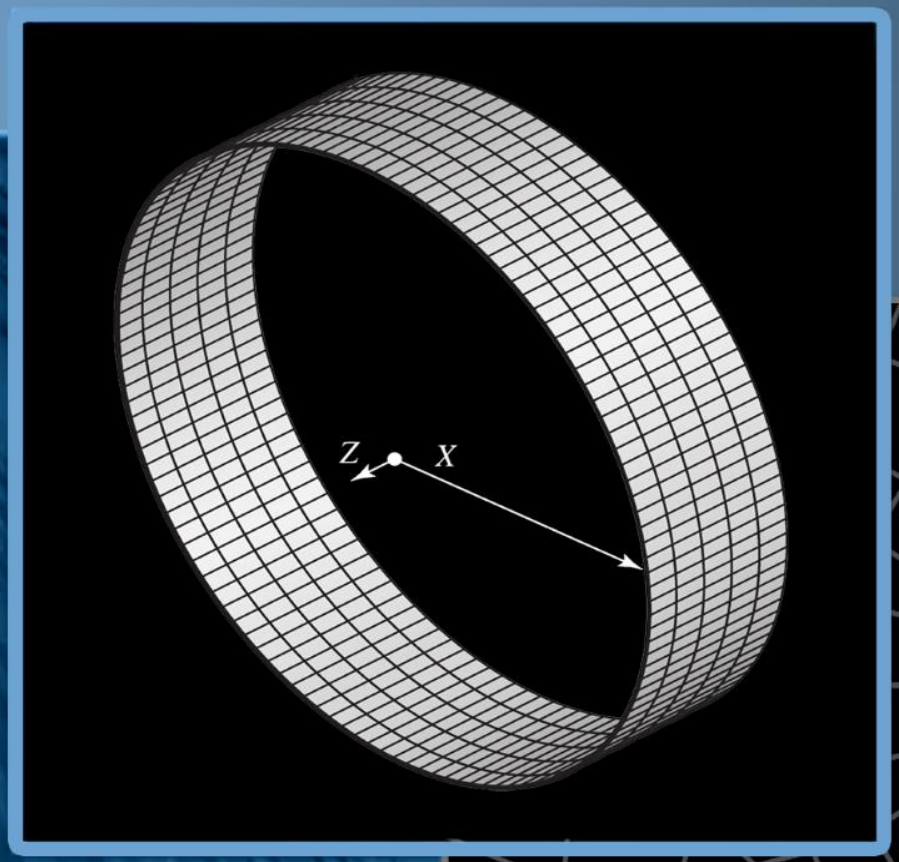
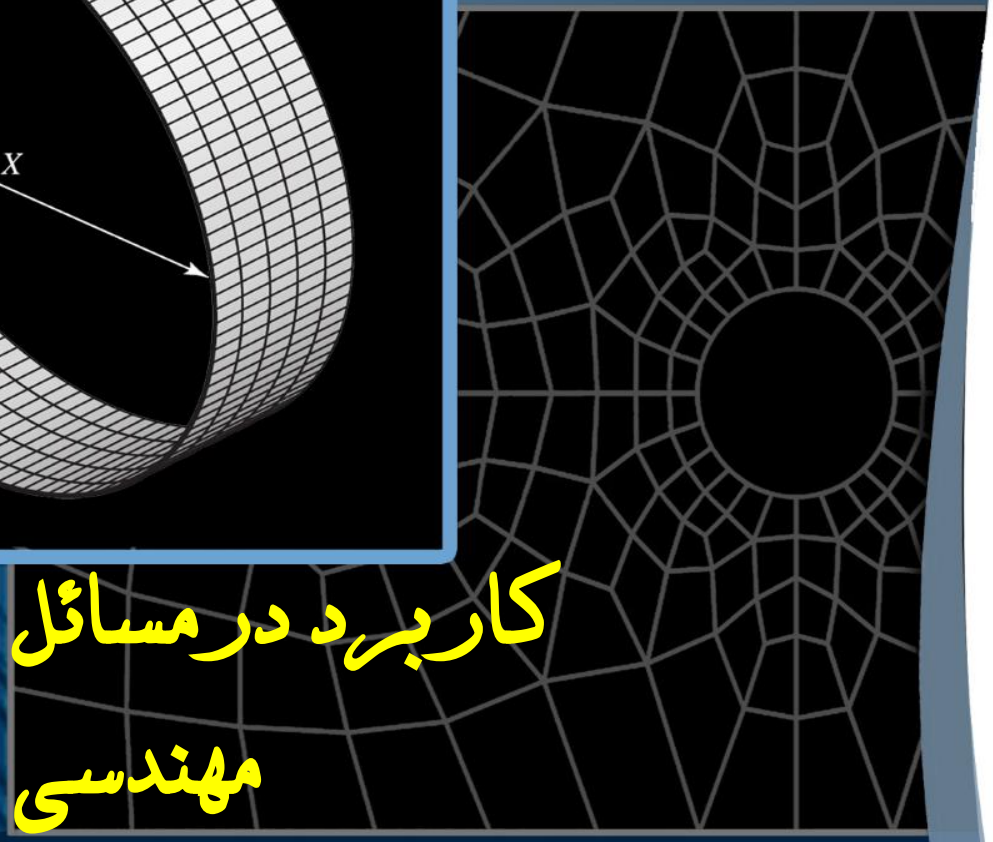
فصل هشتم





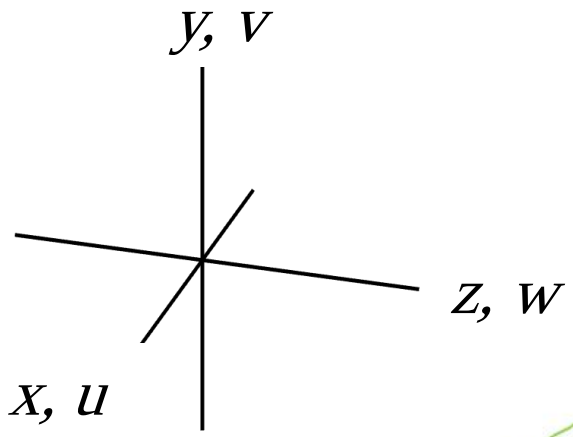
کاربرد درمسائل دو و سه بعدی مهندسی سازه

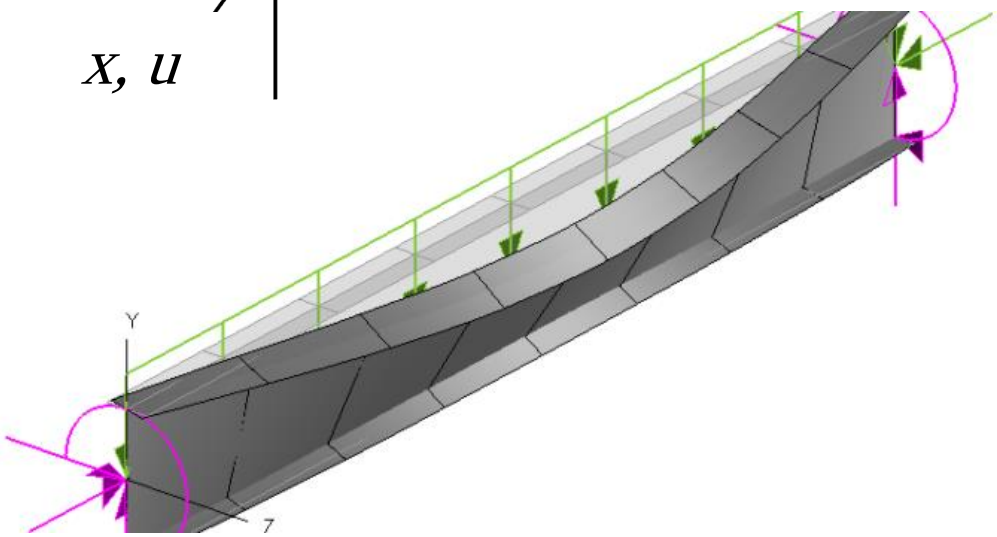
مرجع اصلی: مبانی تحلیل اجزاء محدود، تألیف دیوید هاتن، ترجمه شهاب الدین حاتمی و همکاران

فهرست مطالب

- مفاهیم الاستیسیته
- تنش صفحه‌ای
- کرنش صفحه‌ای
- المان‌های سه‌بعدی
- محاسبه‌ی تنش و کرنش

تغییر مکان نقاط جسم



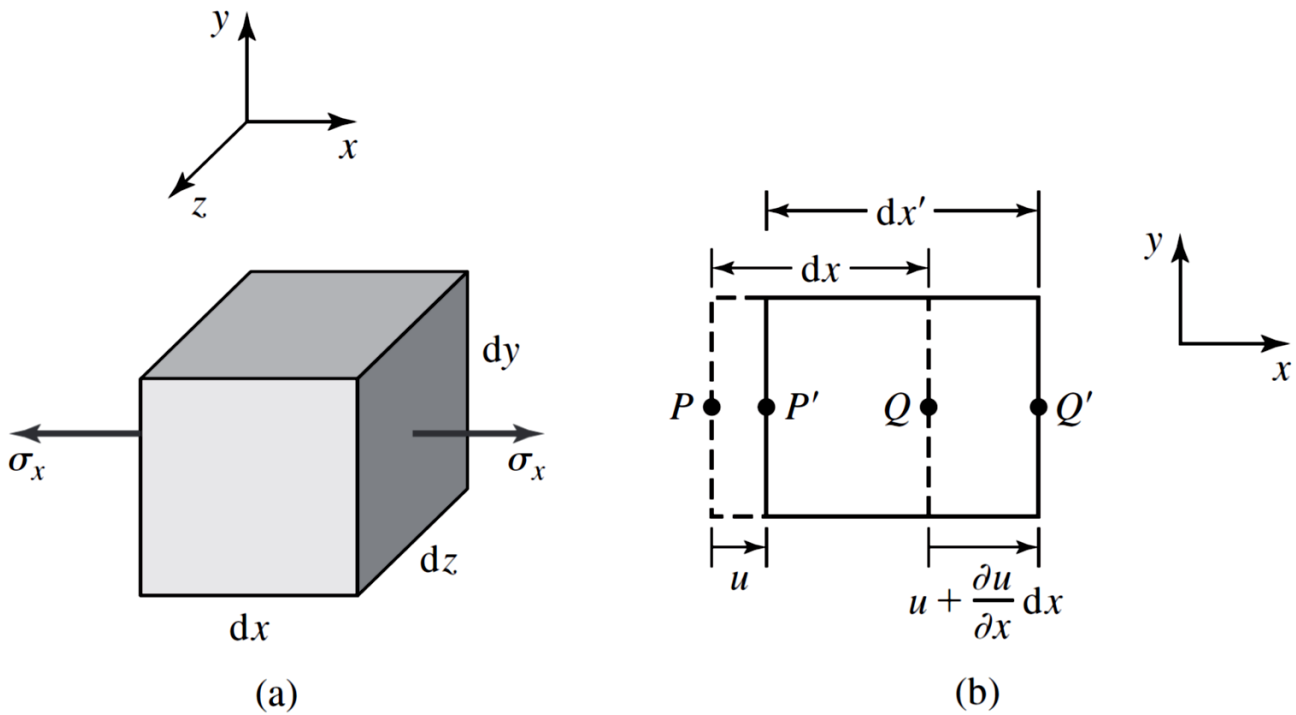


$$u = u(x, y, z)$$

$$v = v(x, y, z)$$

$$w = w(x, y, z)$$

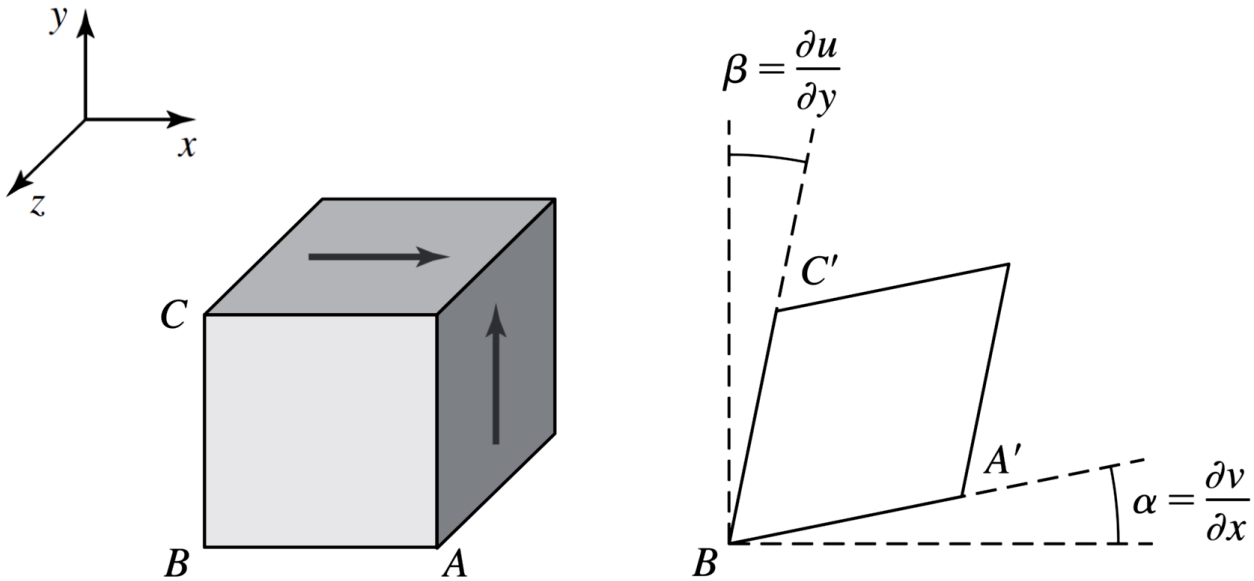
رابطه کرنش- تغییرمکان



$$\varepsilon_x = \frac{dx' - dx}{dx} = \frac{\partial u}{\partial x}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \varepsilon_z = \frac{\partial w}{\partial z}$$

رابطه کرنش- تغییرمکان



(c) (d)

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \quad \text{and} \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

رابطه کرنش- تغییرمکان

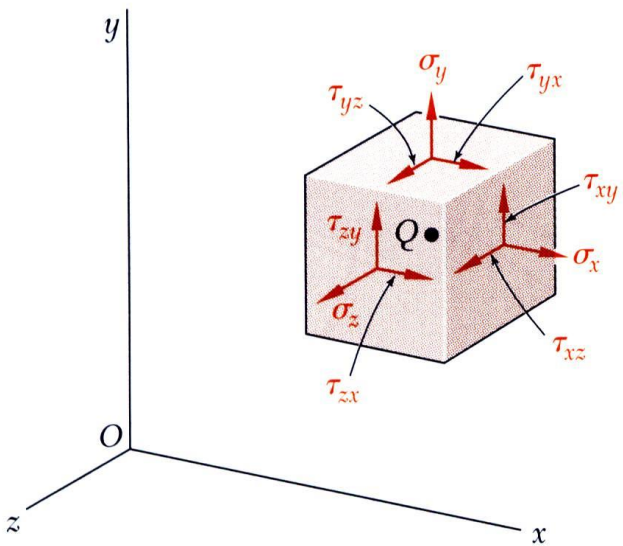
$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial w}{\partial z} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \\ \dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x} \\ \dfrac{\partial v}{\partial z} + \dfrac{\partial w}{\partial y} \end{Bmatrix}$$

معادلات تعمیم یافته برای مواد ایزوتروپیک الاستیک

$$\varepsilon_x = +\frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} - \frac{\nu\sigma_z}{E}$$

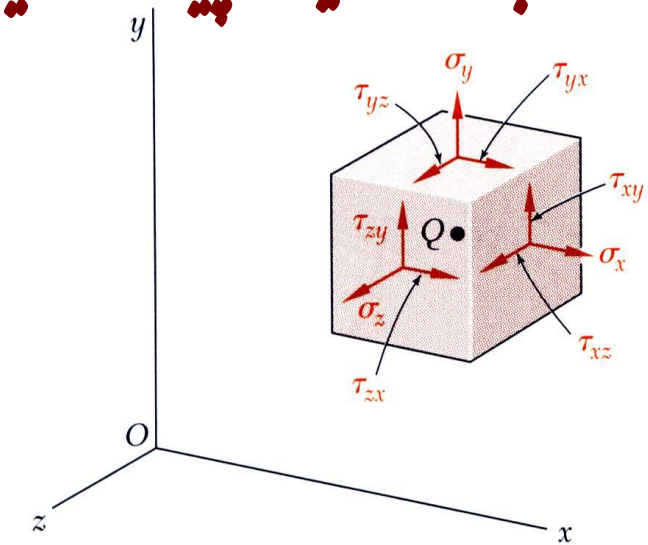
$$\varepsilon_y = -\frac{\nu\sigma_x}{E} + \frac{\sigma_y}{E} - \frac{\nu\sigma_z}{E}$$

$$\varepsilon_z = -\frac{\nu\sigma_x}{E} - \frac{\nu\sigma_y}{E} + \frac{\sigma_z}{E}$$



$$\tau_{xy} = G\gamma_{xy} \quad \tau_{yz} = G\gamma_{yz} \quad \tau_{zx} = G\gamma_{zx}$$

معادلات متقارن برای مواد ایزوتروپیک الاستیک



$$\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{Bmatrix} \qquad \{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix}$$

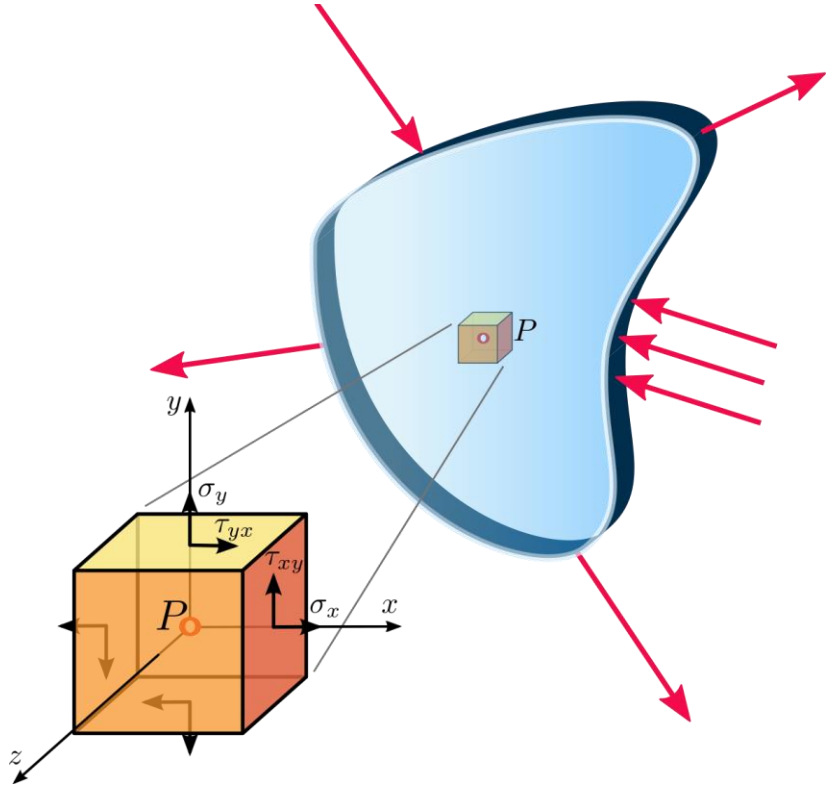
$$= \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \{\varepsilon\} = [D]\{\varepsilon\}$$

تنش صفحه‌ای

$$\sigma_z = \tau_{xz} = \tau_{yz} = 0$$

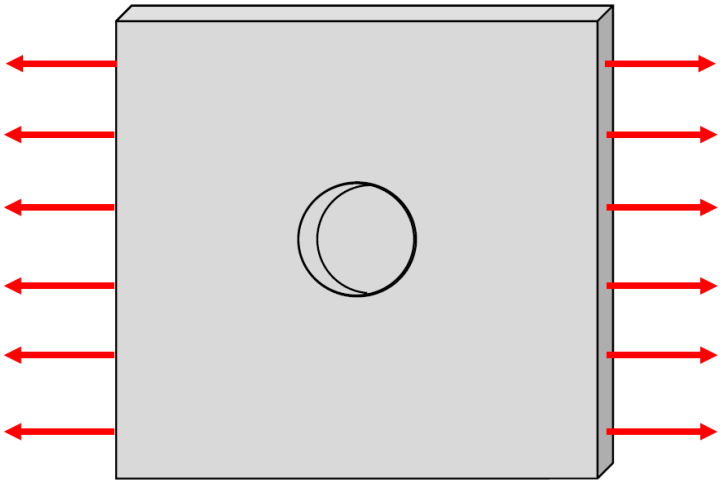
$$\sigma_x = \sigma_x(x,y), \sigma_y = \sigma_y(x,y), \tau_{xy} = \tau_{xy}(x,y)$$

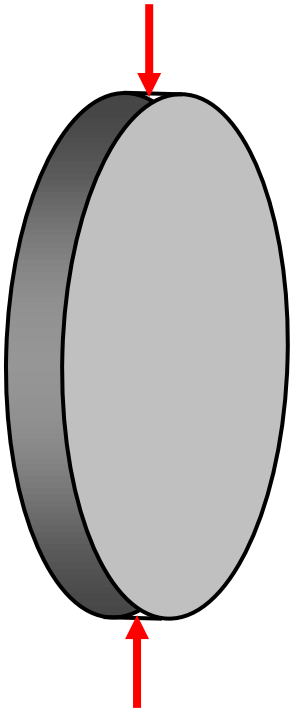
$$(\varepsilon_z \neq 0)$$



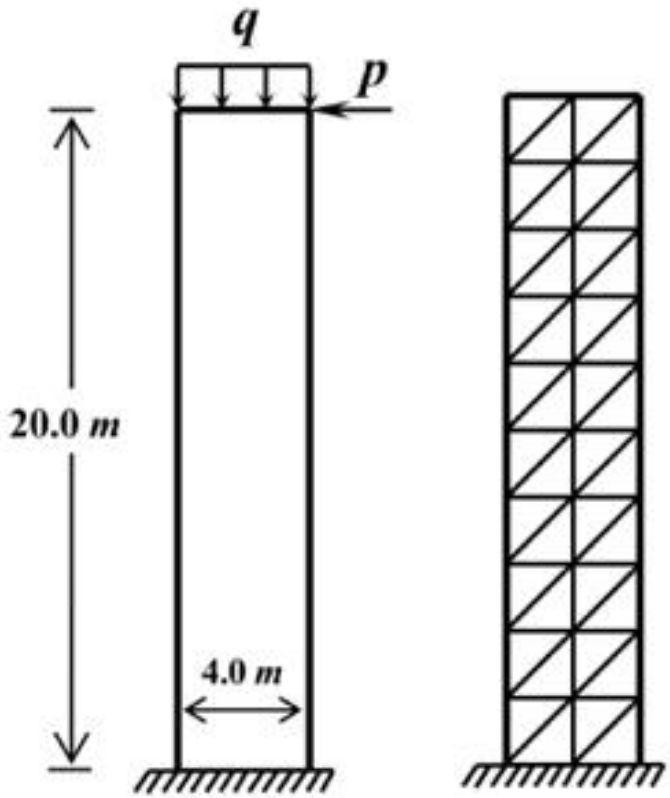
https://en.wikipedia.org/

مثال های تنش صفحه ای

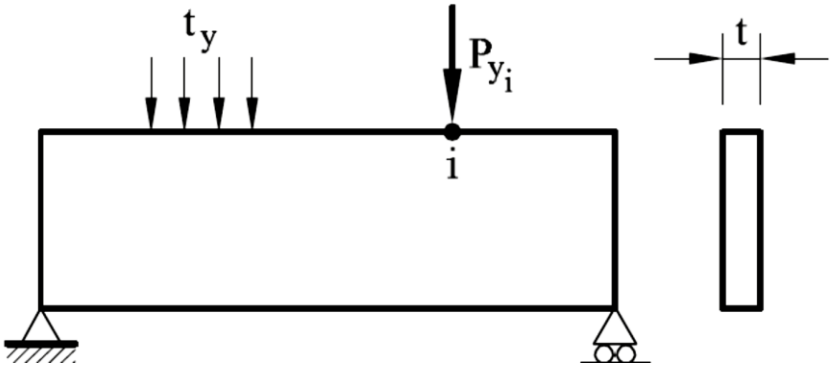




مثال های تنش صفحه ای



دیوار برشی و مدلسازی آن با المان های مثلثی

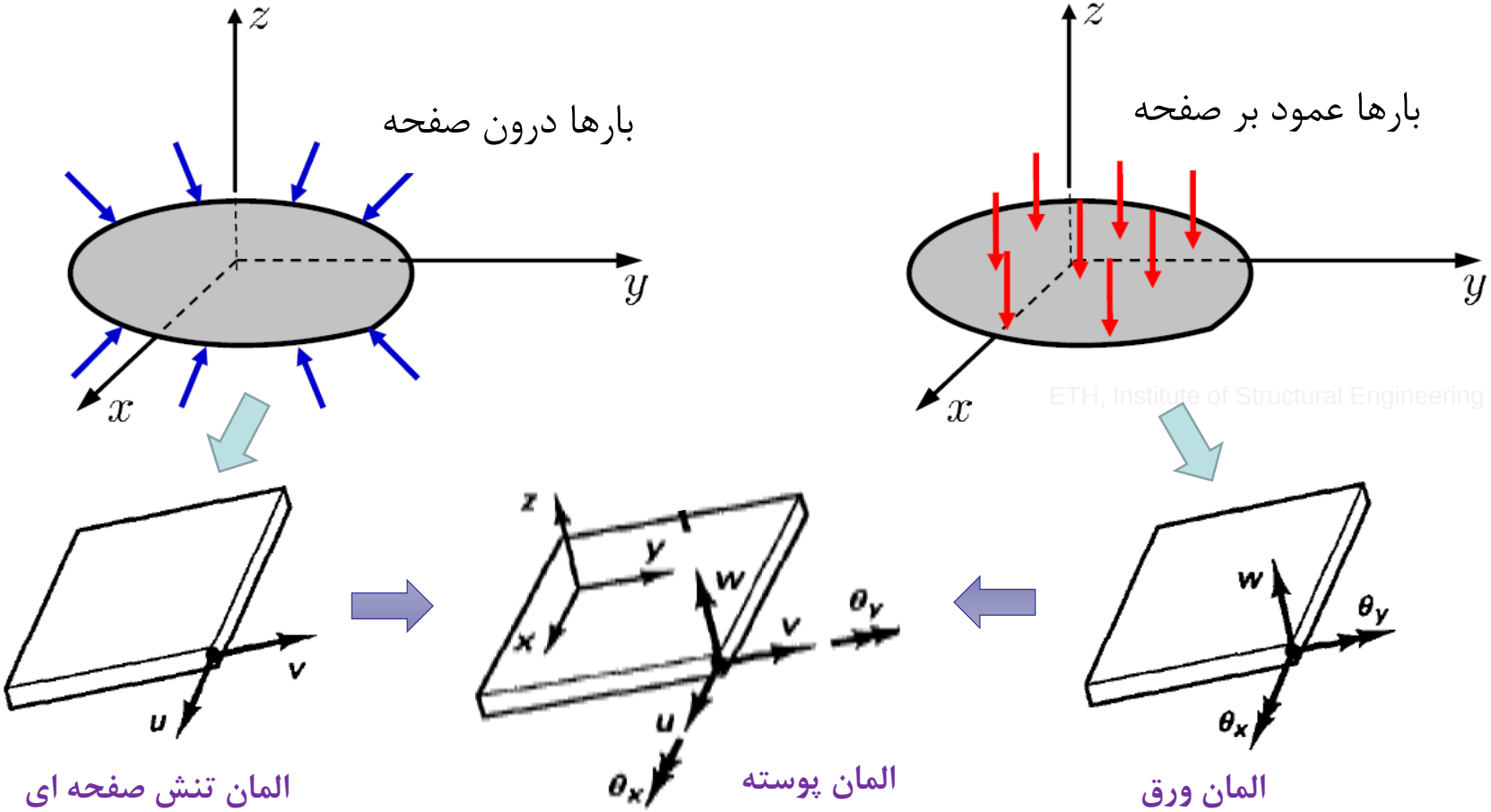


تیر عمیق

مقایسه مسئله تنش صفحه ای با مسئله ورق

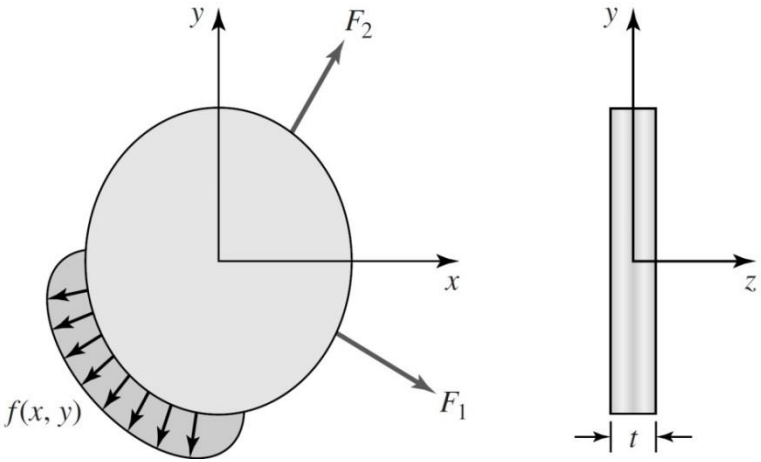
تنش صفحه ای Plane Stress

خمش ورق Plate Bending



درجات آزادی یک گره از المان چهارضلعی

تنش صفحه ای- معادلات متشکله

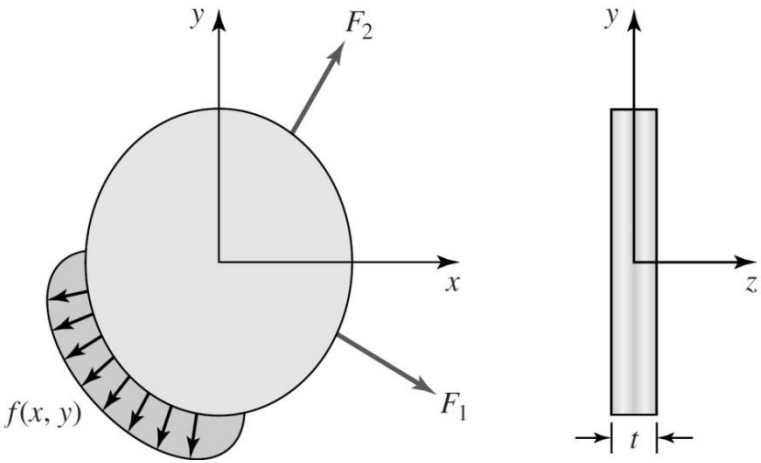


$$\sigma_x = \frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y)$$

$$\sigma_y = \frac{E}{1-\nu^2}(\varepsilon_y + \nu\varepsilon_x)$$

$$\tau_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy} = G\gamma_{xy}$$

تنش صفحه ای- معادلات متشکله



$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

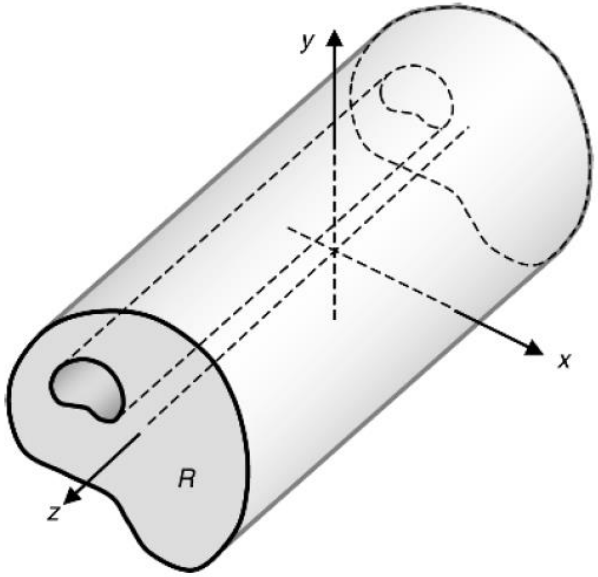
$$\{\sigma\} = [D]\{\varepsilon\}$$

$$[D] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \qquad \{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \qquad \{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

کرنش صفحه‌ای

$$u = u(x, y), v = v(x, y), w = 0$$

$$\varepsilon_z = 0$$
$$\gamma_{xz} = 0$$
$$\gamma_{yz} = 0$$



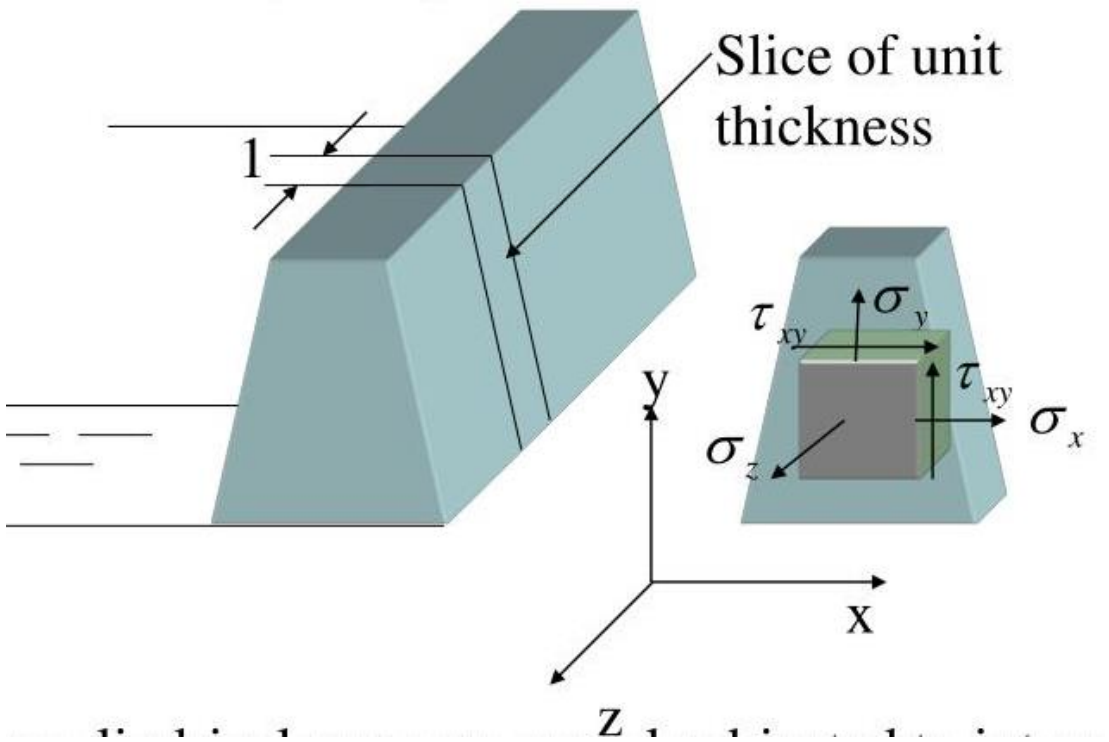
شرایط تحقق کرنش صفحه ای:

(1) سازه منشوری و طول آن از ابعاد دیگر خیلی بزرگ تر باشد.

(2) نیروهای درون صفحه (xy) به جهت طولی (z) وابسته نباشند.

(3) کرنش های نرمال در جهت طولی صفر باشند.

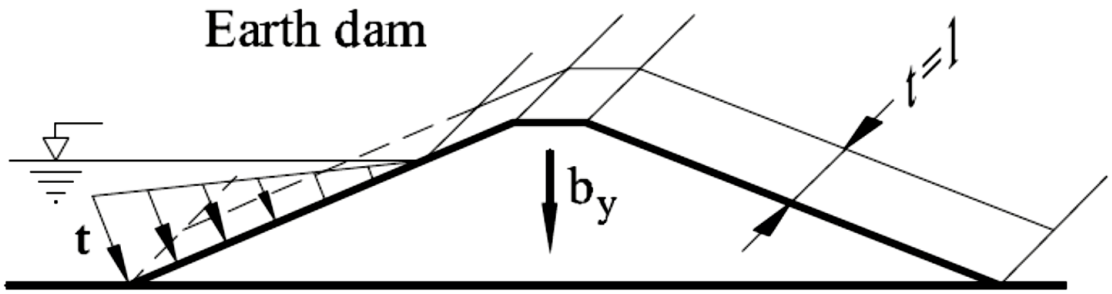
مثال های کرنش صفحه ای

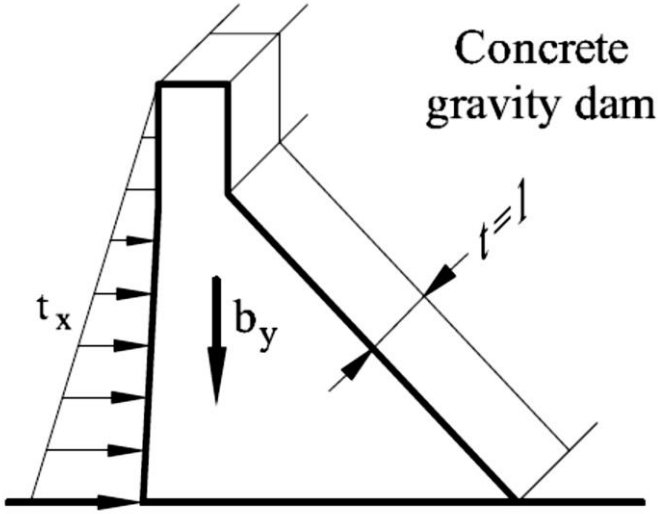


مدلسازی سد بصورت کرنش صفحه ای

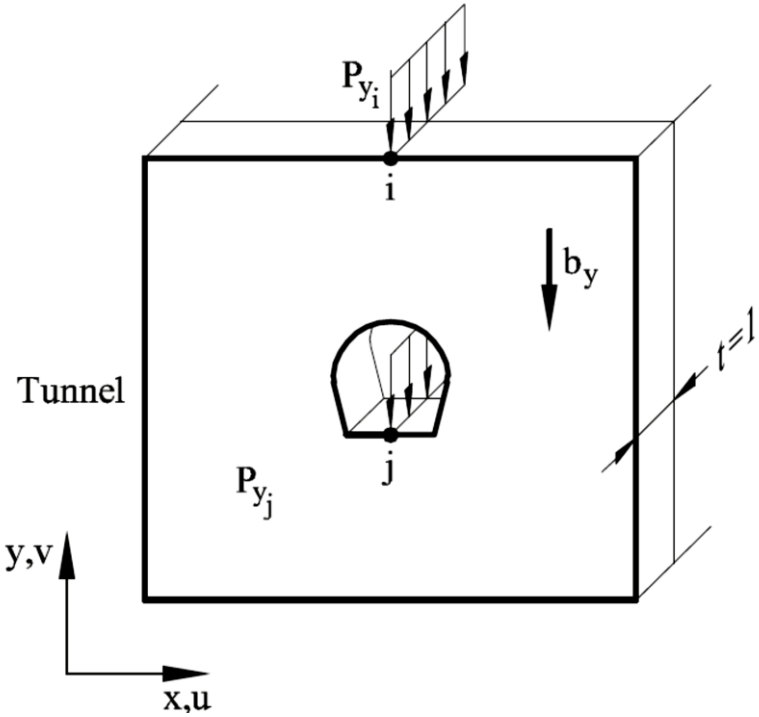
https://i.stack.imgur.com/

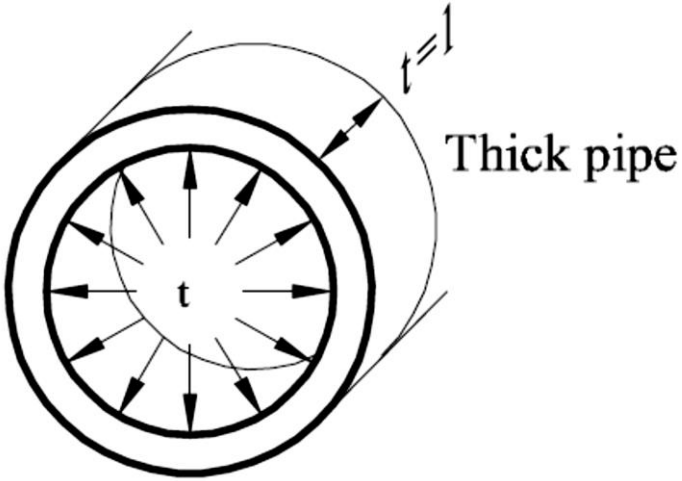
مثال های کرنش صفحه ای





مثال های کرنش صفحه ای

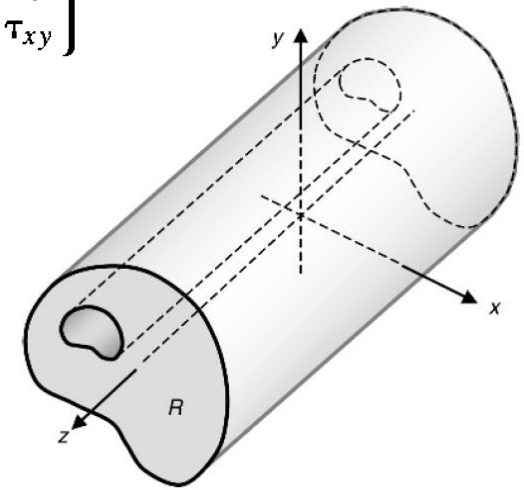




کرنش صفحه ای- معادلات متشکله

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

$$\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$



$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_x + \nu e_y]$$

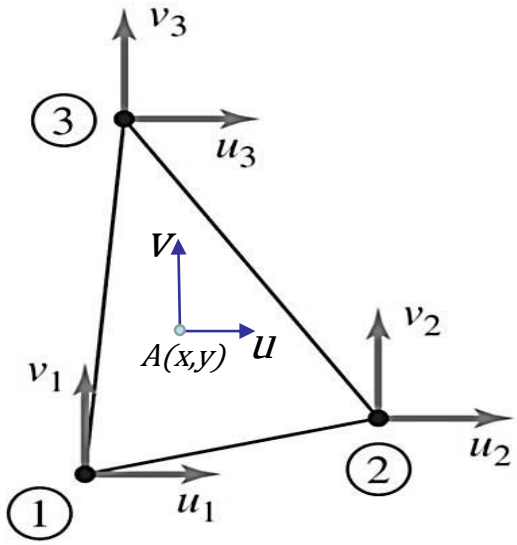
$$\sigma_y = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_y + \nu \varepsilon_x]$$

$$\tau_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy} = G\gamma_{xy}$$

$$\{\sigma\} = [D]\{\varepsilon\}$$

$$[D] = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \dfrac{1-2\nu}{2} \end{bmatrix}$$

المان مثلثی سه گرهی



$$u(x,\ y) = N_1(x,\ y)u_1 + N_2(x,\ y)u_2 + N_3(x,\ y)u_3 = [N]\{u\}$$

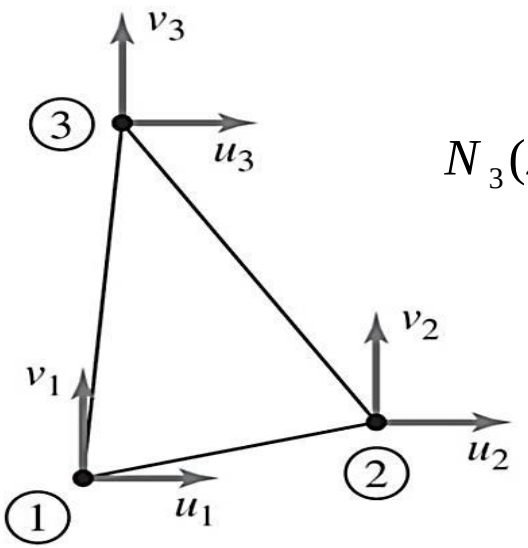
$$v(x,\ y) = N_1(x,\ y)v_1 + N_2(x,\ y)v_2 + N_3(x,\ y)v_3 = [N]\{v\}$$

المان مثلثی سه گرهی

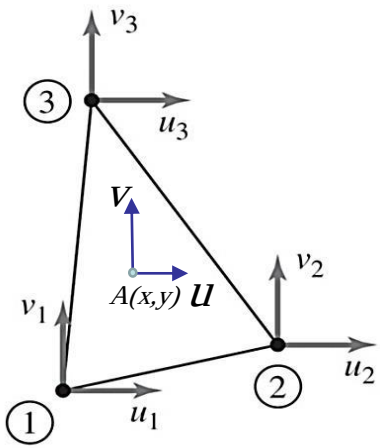
$$N_1(x,y)=\frac{1}{2A}\left[(x_2y_3-x_3y_2)+(y_2-y_3)x+(x_3-x_2)y\right]$$
$$=\frac{1}{2A}(\alpha_1+\beta_1x+\gamma_1y)$$

$$N_2(x,y)=\frac{1}{2A}\left[(x_3y_1-x_1y_3)+(y_3-y_1)x+(x_1-x_3)y\right]$$
$$=\frac{1}{2A}(\alpha_2+\beta_2x+\gamma_2y)$$

$$N_3(x,y)=\frac{1}{2A}\left[(x_1y_2-x_2y_1)+(y_1-y_2)x+(x_2-x_1)y\right]$$
$$=\frac{1}{2A}(\alpha_3+\beta_3x+\gamma_3y)$$



المان مثلثی سه گرهی



$$\varepsilon_x = \frac{\partial u}{\partial x} = \frac{\partial N_1}{\partial x}u_1 + \frac{\partial N_2}{\partial x}u_2 + \frac{\partial N_3}{\partial x}u_3$$

$$\varepsilon_y = \frac{\partial v}{\partial y} = \frac{\partial N_1}{\partial y}v_1 + \frac{\partial N_2}{\partial y}v_2 + \frac{\partial N_3}{\partial y}v_3$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial N_1}{\partial y}u_1 + \frac{\partial N_2}{\partial y}u_2 + \frac{\partial N_3}{\partial y}u_3 + \frac{\partial N_1}{\partial x}v_1 + \frac{\partial N_2}{\partial x}v_2 + \frac{\partial N_3}{\partial x}v_3$$

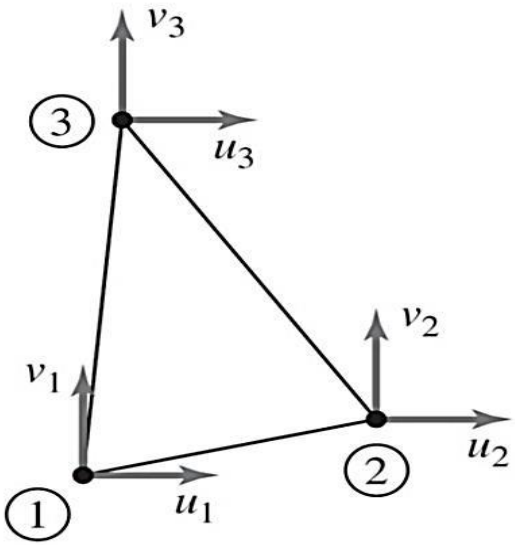
$$\{\varepsilon\} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{Bmatrix} = [B]\{\delta^{(e)}\}$$

المان مثلثی سه گرهی

$$[B]=\frac{1}{2A}\begin{bmatrix} y_2-y_3 & y_3-y_1 & y_1-y_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_3-x_2 & x_1-x_3 & x_2-x_1 \\ x_3-x_2 & x_1-x_3 & x_2-x_1 & y_2-y_3 & y_3-y_1 & y_1-y_2 \end{bmatrix}$$

$$=\frac{1}{2A}\begin{bmatrix} \beta_1 & \beta_2 & \beta_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_1 & \gamma_2 & \gamma_3 & \beta_1 & \beta_2 & \beta_3 \end{bmatrix}$$

با توجه به ثابت بودن کرنش ها در سطح این المان، این المان به عنوان المان مثلثی کرنش ثابت (Constant Strain Triangle یا CST) هم شناخته می شود.

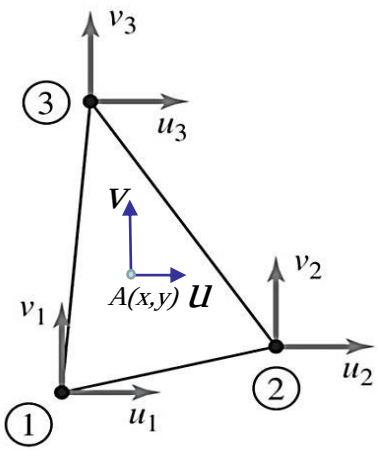


انرژی کرنشی المان

چگالی انرژی کرنشی:

$$u_e = \frac{1}{2}(\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \tau_{xy}\gamma_{xy})$$

$$u_e = \frac{1}{2}\{\varepsilon\}^T\{\sigma\} = \frac{1}{2}\{\varepsilon\}^T[D]\{\varepsilon\}$$



انرژی کرنشی المان محدود:

$$U_e^{(e)} = \frac{1}{2}\iiint_{V^{(e)}} \{\varepsilon\}^T[D]\{\varepsilon\}\, dV^{(e)}$$

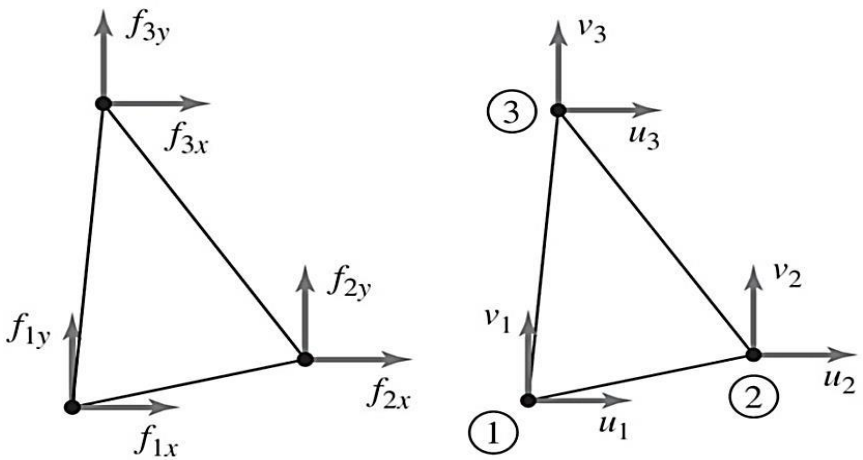
$$U_e^{(e)} = \frac{1}{2}\{\delta^{(e)}\}^T[B]^T[D][B]\{\delta^{(e)}\}\iiint_{V^{(e)}} dV^{(e)}$$

$$= \frac{1}{2}\{\delta^{(e)}\}^T\left(V^{(e)}[B]^T[D][B]\right)\{\delta^{(e)}\}$$

کار نیروهای خارجی المان

$$W = f_{1x}u_1 + f_{2x}u_2 + f_{3x}u_3 + f_{1y}v_1 + f_{2y}v_2 + f_{3y}v_3$$

$$\{f\} = \begin{Bmatrix} f_{1x} \\ f_{2x} \\ f_{3x} \\ f_{1y} \\ f_{2y} \\ f_{3y} \end{Bmatrix}$$



$$W = \{\delta\}^T \{f\}$$

اصل انرژی پتانسیل حداقل

$$\Pi = U_e - W = \frac{V^e}{2}\{\delta\}^T[B]^T[D][B]\{\delta\} - \{\delta\}^T\{f\}$$

$$\frac{\partial \Pi}{\partial \delta_i} = 0 \qquad i = 1, 6$$

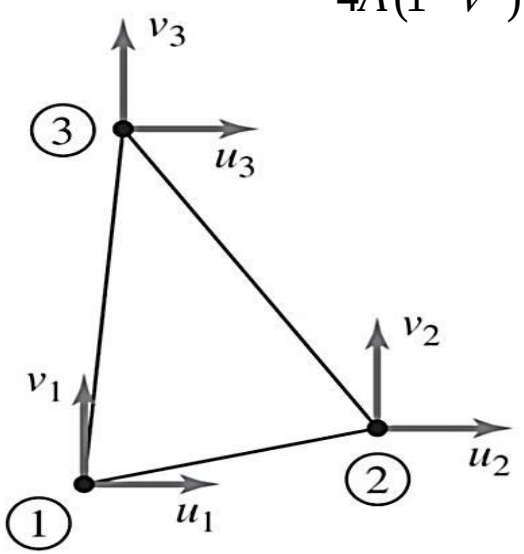
$$V^e[B]^T[D][B]\{\delta\} = \{f\}$$

$$[k]\{\delta\} = \{f\} \qquad \boxed{[k] = V^e[B]^T[D][B]}$$

المان مثلثی سه گرهی- تنش صفحه ای

$$[k]=V^{(e)}[B]^T[D][B]$$

$$[k]=\frac{Et}{4A(1-\nu^2)}\begin{bmatrix}\beta_1 & 0 & \gamma_1\\ \beta_2 & 0 & \gamma_2\\ \beta_3 & 0 & \gamma_3\\ 0 & \gamma_1 & \beta_1\\ 0 & \gamma_2 & \beta_2\\ 0 & \gamma_3 & \beta_3\end{bmatrix}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1-\nu}{2}\end{bmatrix}\begin{bmatrix}\beta_1 & \beta_2 & \beta_3 & 0 & 0 & 0\\ 0 & 0 & 0 & \gamma_1 & \gamma_2 & \gamma_3\\ \gamma_1 & \gamma_2 & \gamma_3 & \beta_1 & \beta_2 & \beta_3\end{bmatrix}$$

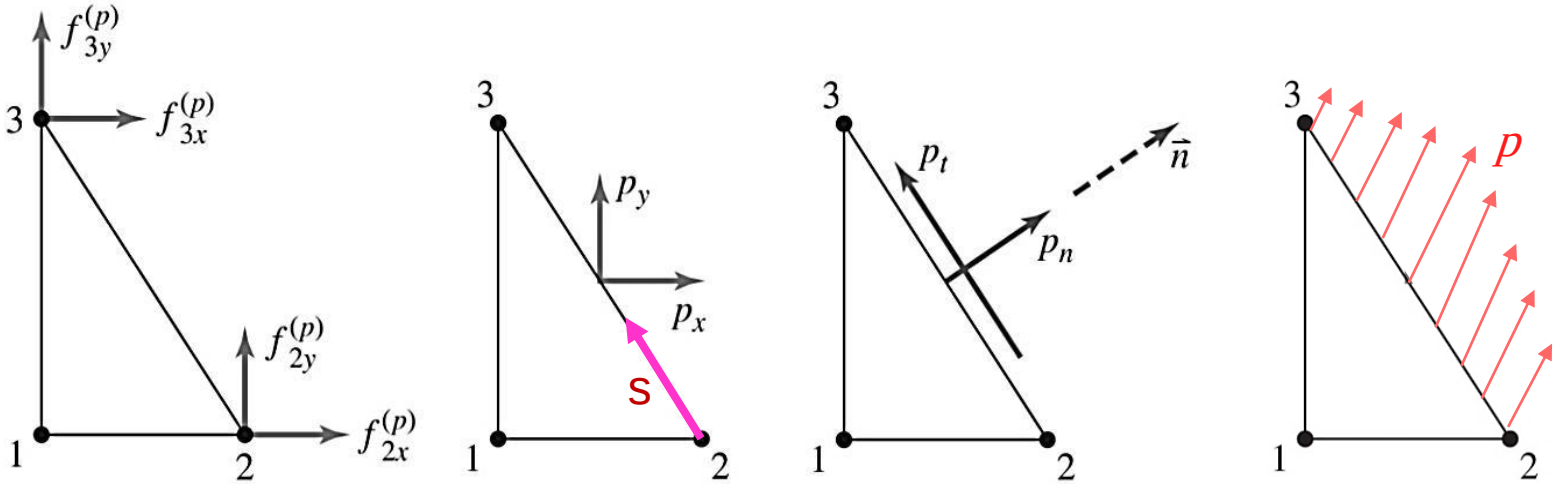


المان مثلثی سه گرهی- تنش صفحه ای

$$[k] = \frac{Et}{4A(1-\nu^2)} \begin{bmatrix} \beta_1^2 + C\gamma_1^2 & \beta_1\beta_2 + C\gamma_1\gamma_2 & \beta_1\beta_3 + C\gamma_1\gamma_3 & \frac{1+\nu}{2}\beta_1\gamma_1 & \nu\beta_1\gamma_2 + C\beta_2\gamma_1 & \nu\beta_1\gamma_3 + C\beta_3\gamma_1 \\ & \beta_2^2 + C\gamma_2^2 & \beta_2\beta_3 + C\gamma_2\gamma_3 & \nu\beta_2\gamma_1 + C\beta_1\gamma_2 & \frac{1+\nu}{2}\beta_2\gamma_2 & \nu\beta_2\gamma_3 + C\beta_3\gamma_2 \\ & & \beta_3^2 + C\gamma_3^2 & \nu\beta_3\gamma_1 + C\beta_1\gamma_3 & \nu\beta_3\gamma_2 + C\beta_2\gamma_3 & \frac{1+\nu}{2}\beta_3\gamma_3 \\ & SYM & & \gamma_1^2 + C\beta_1^2 & \gamma_1\gamma_2 + C\beta_1\beta_2 & \gamma_1\gamma_3 + C\beta_1\beta_3 \\ & & & & \gamma_2^2 + C\beta_2^2 & \gamma_2\gamma_3 + C\beta_2\beta_3 \\ & & & & & \gamma_3^2 + C\beta_3^2 \end{bmatrix}$$

$$C = (1-\nu)/2$$

بارگیری معادل بار گسترده در لبه



$$p_x = p_n n_x - p_t n_y$$

$$p_y = p_n n_y + p_t n_x$$

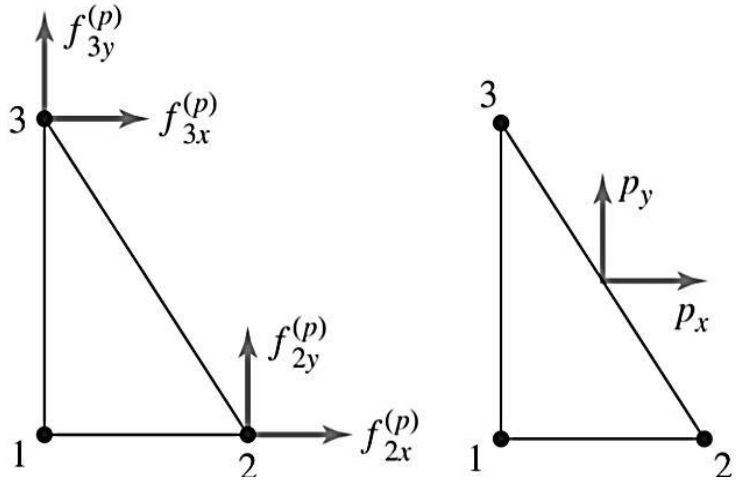
$$W_p = t \int_2^3 p_x u(x, y)\, \mathrm{d}S + t \int_2^3 p_y v(x, y)\, \mathrm{d}S$$

$$u(x, y) = N_2(x, y)u_2 + N_3(x, y)u_3$$

$$v(x, y) = N_2(x, y)v_2 + N_3(x, y)v_3$$

N_1 در امتداد ضلع ۳-۲ صفر است:

بارگرهی معادل بار گسترده در لبه



$$W_p = t \int_2^3 p_x[N_2(x, y)u_2 + N_3(x, y)u_3]\,\mathrm{d}S$$

$$+ t \int_2^3 p_y[N_2(x, y)v_2 + N_3(x, y)v_3]\,\mathrm{d}S$$

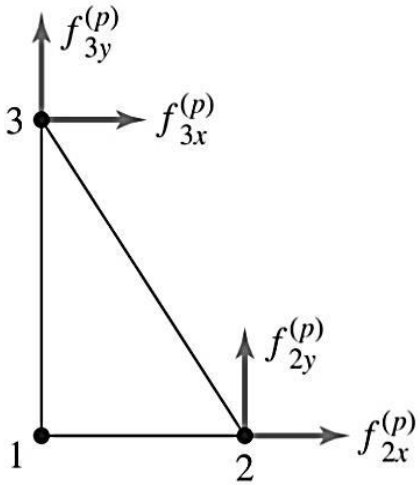
$$W_p = f_{2x}^{(p)}u_2 + f_{3x}^{(p)}u_3 + f_{2y}^{(p)}v_2 + f_{3y}^{(p)}v_3$$

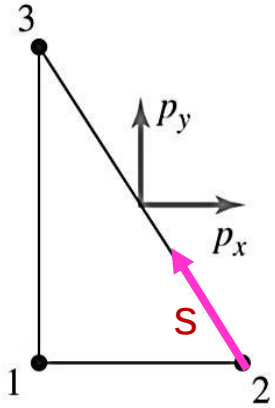
بارگیری معادل بار گسترده در لبه

$$f_{2x}^{(p)} = t\int_{2}^{3} p_x N_2(x,y)\,\mathrm{d}S$$

$$f_{3x}^{(p)} = t\int_{2}^{3} p_x N_3(x,y)\,\mathrm{d}S$$

$$[N]^T = \begin{bmatrix} N_1 & 0 \\ N_2 & 0 \\ N_3 & 0 \\ 0 & N_1 \\ 0 & N_2 \\ 0 & N_3 \end{bmatrix} \qquad \{f^{(p)}\} = \begin{Bmatrix} f_{1x}^{(p)} \\ f_{2x}^{(p)} \\ f_{3x}^{(p)} \\ f_{1y}^{(p)} \\ f_{2y}^{(p)} \\ f_{3y}^{(p)} \end{Bmatrix}$$

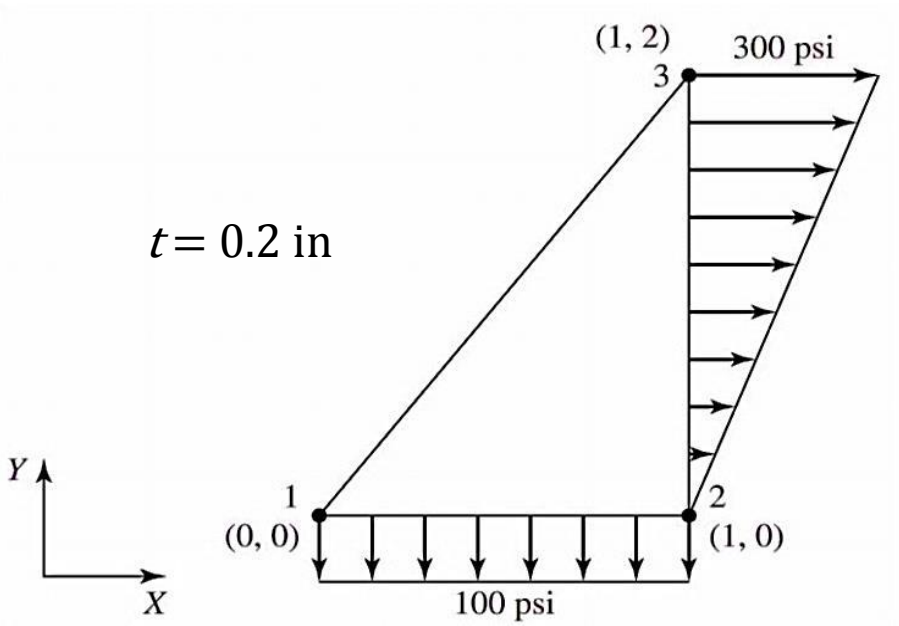




$$\{f^{(p)}\} = \int_S [N]^T \begin{Bmatrix} p_x \\ p_y \end{Bmatrix} t\,dS$$

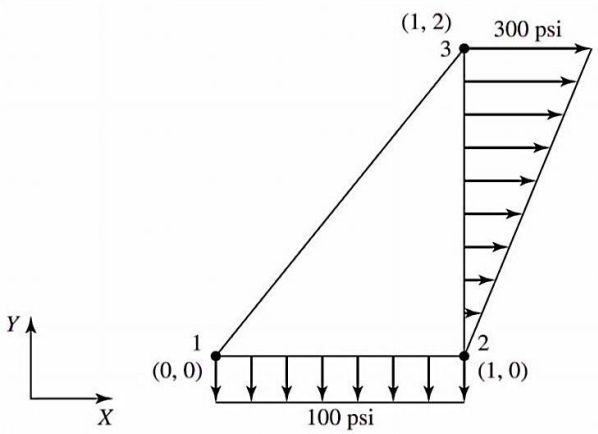
مثال

برای المان زیر، بار گرهی معادل بار گسترده در لبه ها را محاسبه کنید.



$$\{f^{(p)}\} = \int_S [N]^T \begin{Bmatrix} p_x \\ p_y \end{Bmatrix} t dS$$

مثال



$$N_1(x,y)=\frac{1}{2A}\left[(x_2y_3-x_3y_2)+(y_2-y_3)x+(x_3-x_2)y\right]$$

$$N_2(x,y)=\frac{1}{2A}\left[(x_3y_1-x_1y_3)+(y_3-y_1)x+(x_1-x_3)y\right]$$

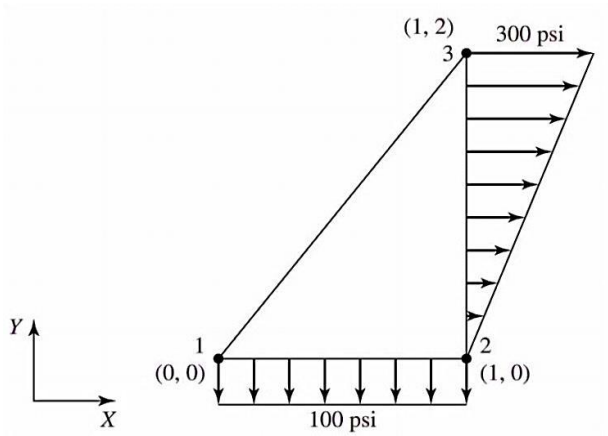
$$N_3(x,y)=\frac{1}{2A}\left[(x_1y_2-x_2y_1)+(y_1-y_2)x+(x_2-x_1)y\right]$$

با استفاده از مختصات گرهی تعیین شده، توابع درونیاب (با مساحت المان A=1 عبارتند از:

$$N_1(x,y)=\frac{1}{2}\left[2-2x\right]=1-x$$

$$N_2(x,y)=\frac{1}{2}\left[2x-y\right]$$

$$N_3(x,y)=\frac{1}{2}y$$



$$N_1(x,y)=\frac{1}{2}[2-2x]=1-x$$

$$N_2(x,y)=\frac{1}{2}[2x-y]$$

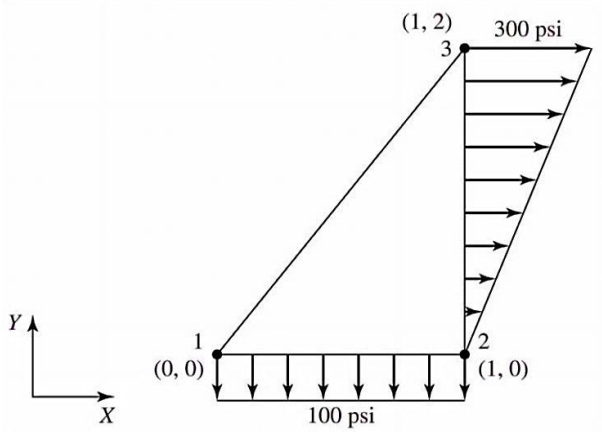
$$N_3(x,y)=\frac{1}{2}y$$

مثال

در امتداد ضلع ۱-۲:

$$p_y=-100 \quad psi \qquad p_x=0 \qquad y=0$$

$$\{f^{(p)}\}=\int_S \begin{bmatrix} 1-x & 0 \\ x & 0 \\ 0 & 0 \\ 0 & 1-x \\ 0 & x \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ -100 \end{Bmatrix} t\,dS \quad =0.2\int_0^1 \begin{bmatrix} 1-x & 0 \\ x & 0 \\ 0 & 0 \\ 0 & 1-x \\ 0 & x \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ -100 \end{Bmatrix} dx = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -10 \\ -10 \\ 0 \end{Bmatrix} lb$$



مثال

$$N_1(x,y)=\frac{1}{2}[2-2x]=1-x$$

$$N_2(x,y)=\frac{1}{2}[2x-y]$$

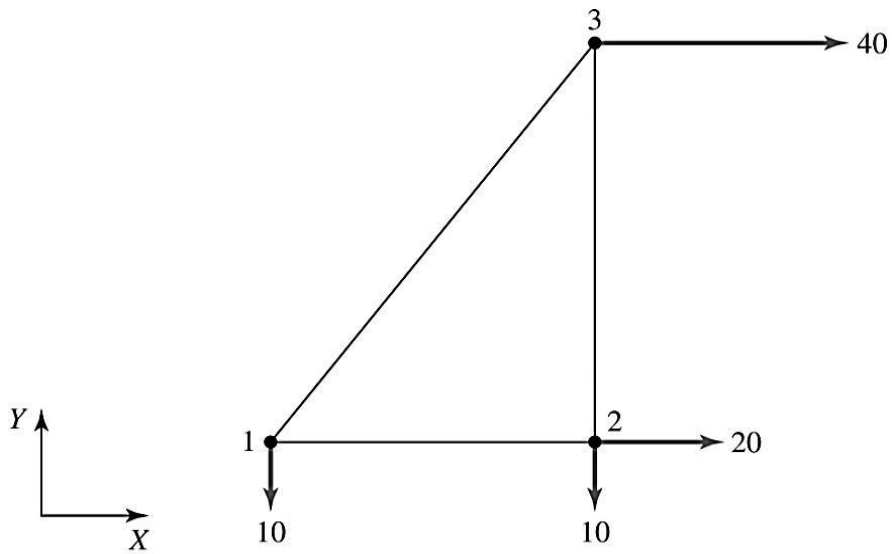
$$N_3(x,y)=\frac{1}{2}y$$

در امتداد ضلع ۳-۲:

$$x=1 \qquad p_x=150y \qquad p_y=0$$

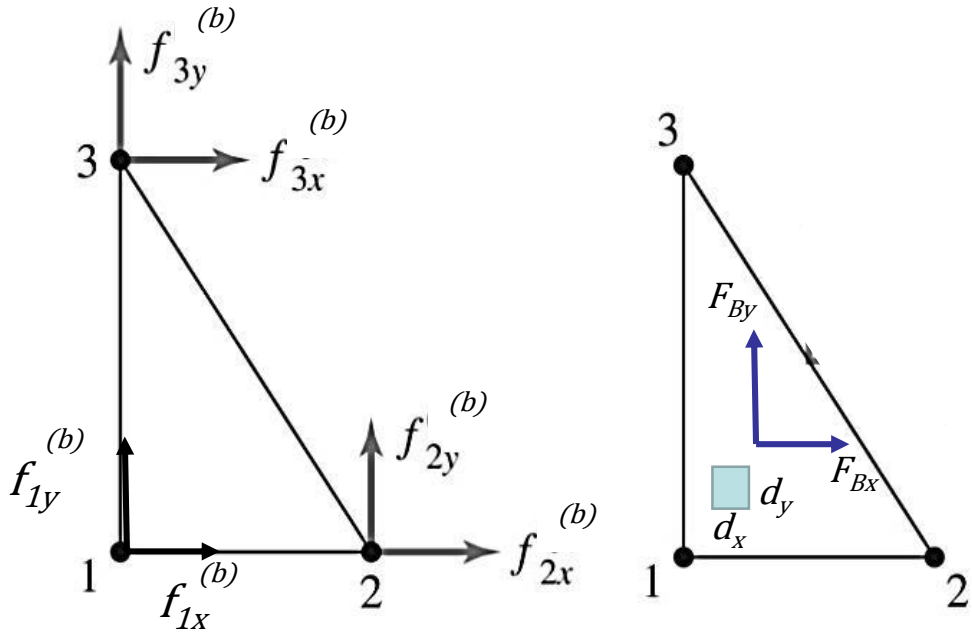
$$\{f^{(p)}\}=\int_S \begin{bmatrix} 0 & 0 \\ \frac{1}{2}(2-y) & 0 \\ \frac{y}{2} & 0 \\ 0 & 0 \\ 0 & \frac{1}{2}(2-y) \\ 0 & \frac{y}{2} \end{bmatrix} \begin{Bmatrix} 150y \\ 0 \end{Bmatrix} t\,dS = 0.2\int_0^2 \begin{bmatrix} 0 & 0 \\ \frac{1}{2}(2-y) & 0 \\ \frac{y}{2} & 0 \\ 0 & 0 \\ 0 & \frac{1}{2}(2-y) \\ 0 & \frac{y}{2} \end{bmatrix} \begin{Bmatrix} 150y \\ 0 \end{Bmatrix} dy = \begin{Bmatrix} 0 \\ 20 \\ 40 \\ 0 \\ 0 \\ 0 \end{Bmatrix} lb$$

مثال



$$\left\{f^{(p)}\right\} = \begin{Bmatrix} f_{1x} \\ f_{2x} \\ f_{3x} \\ f_{1y} \\ f_{2y} \\ f_{3y} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 20 \\ 40 \\ -10 \\ -10 \\ 0 \end{Bmatrix} lb$$

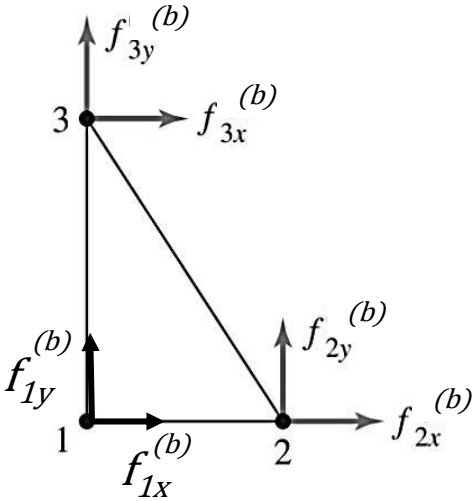
بارگیری معادل نیروی حجمی

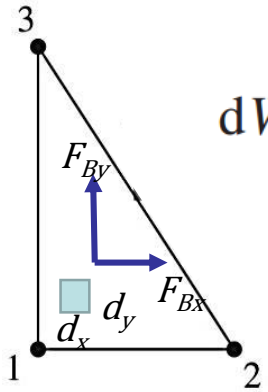


نیروهای وارد بر واحد جرم جسم در جهات مختصات: $\begin{Bmatrix} F_{BX} \\ F_{BY} \end{Bmatrix}$

جرم المان بینهایت کوچک: $\rho\, t\, d_x\, d_y$

بارگیری معادل نیروی حجمی





$$\mathrm{d}W_b = \rho\, F_{BX} u t\, \mathrm{d}x\, \mathrm{d}y + \rho\, F_{BY} v t\, \mathrm{d}x\, \mathrm{d}y$$

$$W_b = \rho t \iint_A F_{BX}(N_1 u_1 + N_2 u_2 + N_3 u_3)\, \mathrm{d}x\, \mathrm{d}y$$

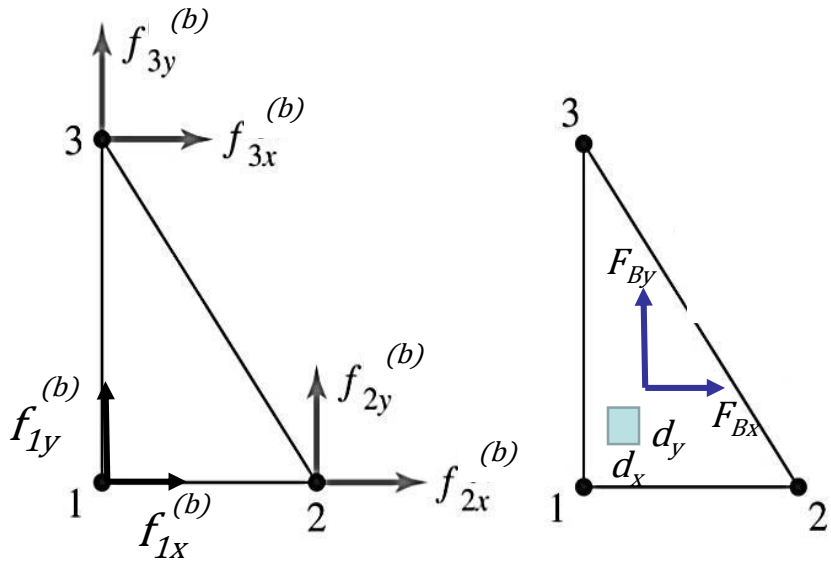
$$+ \rho t \iint_A F_{BY}(N_1 v_1 + N_2 v_2 + N_3 v_3)\, \mathrm{d}x\, \mathrm{d}y$$

$$W_b = f_{1x}^{(b)} u_1 + f_{2x}^{(b)} u_2 + f_{3x}^{(b)} u_3 + f_{1y}^{(b)} v_1 + f_{2y}^{(b)} v_2 + f_{3y}^{(b)} v_3$$

بارگیری معادل نیروی حجمی

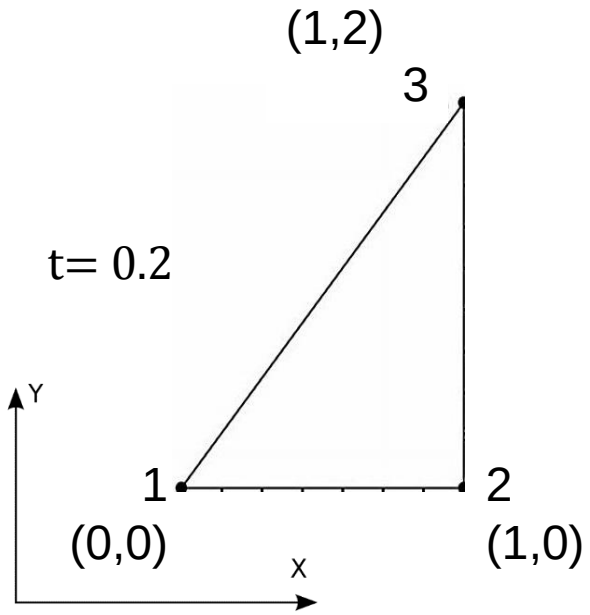
$$f_{ix}^{(b)} = \rho t \int_A N_i F_{BX}\, dx\, dy \qquad i = 1, 3$$

$$f_{iy}^{(b)} = \rho t \int_A N_i F_{BY}\, dx\, dy \qquad i = 1, 3$$



$$[N]^T = \begin{bmatrix} N_1 & 0 \\ N_2 & 0 \\ N_3 & 0 \\ 0 & N_1 \\ 0 & N_2 \\ 0 & N_3 \end{bmatrix}$$

$$\{f^{(b)}\} = \rho t \int_A [N]^T \begin{Bmatrix} F_{BX} \\ F_{BY} \end{Bmatrix} dx\, dy$$



مثال

مولفه‌های نیروهای گرهی ناشی از نیروی وزن را برای المان نشان‌داده‌شده تعیین کنید.

$$\rho = 7.3\times10^{-4} \quad \text{slug/in}^3$$

$$\begin{Bmatrix} F_{BX} \\ F_{BY} \end{Bmatrix} = \begin{Bmatrix} 0 \\ -386.4 \end{Bmatrix} \; in/\sec^2$$

با استفاده از مختصات گرهی تعیین شده، توابع درونیاب (با مساحت المان A=1 عبارتند از:

$$N_1(x,y) = \frac{1}{2}[2-2x] = 1-x$$

$$N_2(x,y) = \frac{1}{2}[2x-y]$$

$$N_3(x,y) = \frac{1}{2}y$$

مثال

$$f_{iy}^{(b)} = \rho t \int_A N_i F_{BY} \, dx \, dy \qquad i = 1, 3$$

$$f_{1y}^{(b)} = \rho t \iint_A F_{BY} N_1 \, dx \, dy = \rho t \iint_A F_{BY} (1 - x) dx \, dy$$

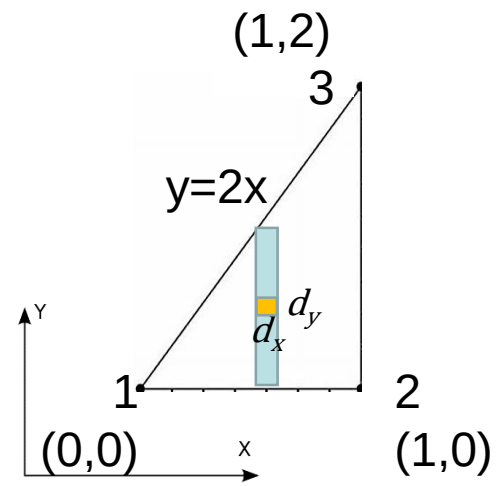
$$f_{2y}^{(b)} = \rho t \iint_A F_{BY} N_2 \, dx \, dy = \rho t \iint_A \frac{F_{BY}}{2} (2x - y) dx \, dy$$

$$f_{3y}^{(b)} = \rho t \iint_A F_{BY} N_3 \, dx \, dy = \rho t \iint_A \frac{F_{BY}}{2} y \, dx \, dy$$

مثال

$$f_{1y}^{(b)} = \rho t F_{BY} \int_0^1 \int_0^{2x} (1-x)\,dy\,dx = \rho t F_{BY} \int_0^1 2x(1-x)\,dx = \rho t Y \left(x^2 - \frac{2x^3}{3} \right)\Bigg|_0^1$$

$$= \frac{\rho t F_{BY}}{3} = -0.0189 \text{ lb}$$



$$f_{2y}^{(b)} = \rho t F_{BY} \int_0^1 \int_0^{2x} \frac{1}{2}(2x - y)\,dy\,dx = \frac{\rho t F_{BY}}{2} \int_0^1 2x^2\,dx$$

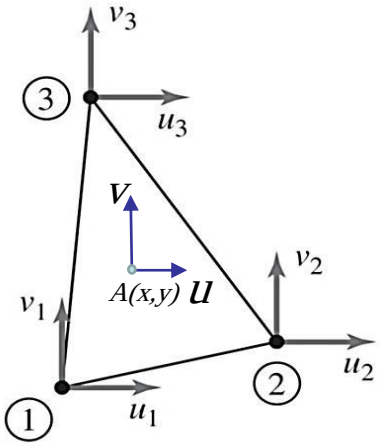
$$= \frac{\rho t F_{BY}}{2}\left(\frac{2}{3}\right) = \frac{\rho t F_{BY}}{3} = -0.0189 \text{ lb}$$

$$f_{3y}^{(b)} = \frac{\rho t F_{BY}}{2} \int_0^1 \int_0^{2x} y\,dx\,dy = \frac{\rho t F_{BY}}{2} \int_0^1 2x^2\,dx = \frac{\rho t F_{BY}}{3} = -0.0189 \text{ lb}$$

انرژی پتانسیل المان کرنش صفحه‌ای

مشابه تنش صفحه‌ای می توان نوشت:

$$U_e = \frac{1}{2}\iiint_V (\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \tau_{xy}\gamma_{xy})\, dV$$



$$U_e = \frac{1}{2}\iiint_V [\varepsilon_x \quad \varepsilon_y \quad \gamma_{xy}] \frac{E}{(1+\nu)(1-2\nu)}$$

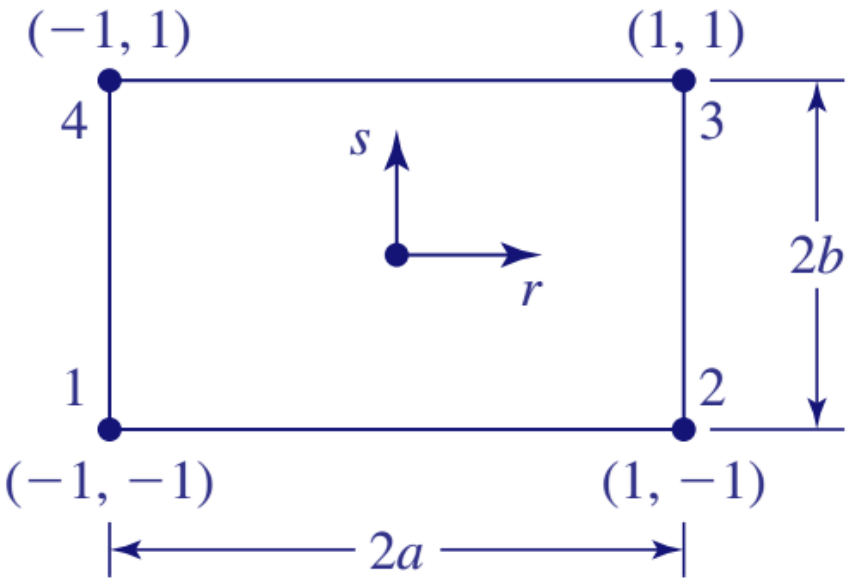
$$\times \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} dV$$

با این تفاوت که ماتریس خواص الاستیک در حالت کرنش صفحه‌ای به‌صورت زیر تعریف می‌شود:

$$[D] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

$$\boxed{[k] = V^e [B]^T [D][B]}$$

المان مستطیلی چهارگرهی



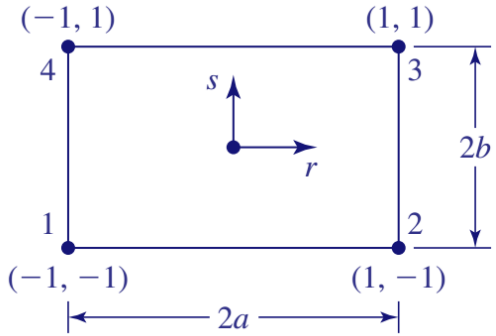
یک المان مستطیلی به عرض $2a$ و ارتفاع $2b$

المان مستطیلی چهارگرهی

$$\varepsilon_x = \frac{\partial u}{\partial x}$$

$$\varepsilon_y = \frac{\partial v}{\partial y}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$



$$N_1(r,s) = \frac{1}{4}(1-r)(1-s)$$

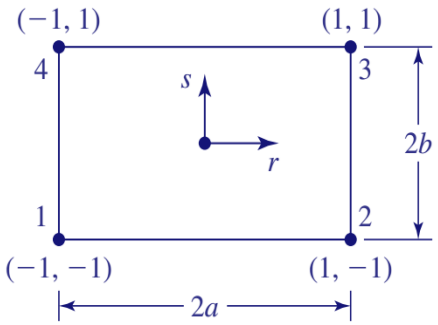
$$N_2(r,s) = \frac{1}{4}(1+r)(1-s)$$

$$N_3(r,s) = \frac{1}{4}(1+r)(1+s)$$

$$N_4(r,s) = \frac{1}{4}(1-r)(1+s)$$

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \overbrace{\begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial x} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial y} & \frac{\partial N_4}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial y} & \frac{\partial N_4}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial x} \end{bmatrix}}^{[B]} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{Bmatrix}$$

المان مستطیلی چهارگرهی



مشتق گیری زنجیره ای:

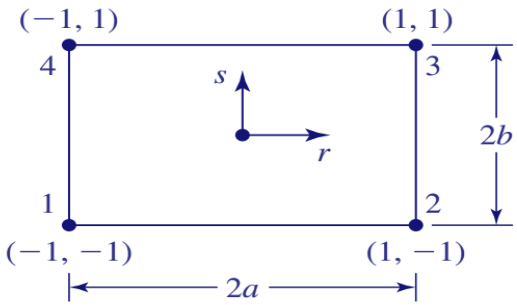
$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r}\frac{\partial r}{\partial x} = \frac{1}{a}\frac{\partial}{\partial r}$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial s}\frac{\partial s}{\partial y} = \frac{1}{b}\frac{\partial}{\partial s}$$

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

$$= \underbrace{\begin{bmatrix} \frac{s-1}{4a} & \frac{1-s}{4a} & \frac{1+s}{4a} & -\frac{1+s}{4a} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{r-1}{4b} & -\frac{1+r}{4b} & \frac{1+r}{4b} & \frac{1-r}{4b} \\ \frac{r-1}{4b} & -\frac{1+r}{4b} & \frac{1+r}{4b} & \frac{1-r}{4b} & \frac{s-1}{4a} & \frac{1-s}{4a} & \frac{1+s}{4a} & -\frac{1+s}{4a} \end{bmatrix}}_{[B]} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{Bmatrix}$$

المان مستطیلی چهارگرهی، کرنش صفحه‌ای

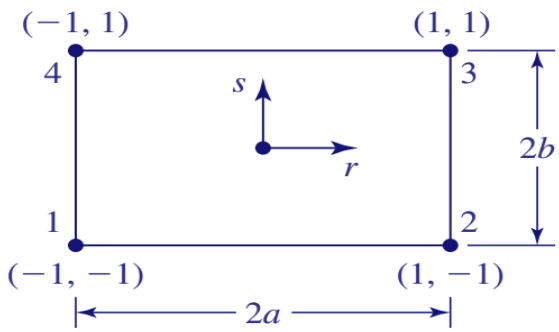


$$[k^{(e)}] = \iiint_{V^{(e)}} [B^T][D][B]\,\mathrm{d}V^{(e)}$$

$$= \frac{Etab}{(1+\nu)(1-2\nu)} \int_{-1}^{1}\int_{-1}^{1} \begin{bmatrix} \frac{s-1}{4a} & 0 & \frac{r-1}{4b} \\ \frac{1-s}{4a} & 0 & -\frac{1+r}{4b} \\ \frac{1+s}{4a} & 0 & \frac{1+r}{4b} \\ -\frac{1+s}{4a} & 0 & \frac{1-r}{4b} \\ 0 & \frac{r-1}{4b} & \frac{s-1}{4a} \\ 0 & -\frac{1+r}{4b} & \frac{1-s}{4a} \\ 0 & \frac{1+r}{4b} & \frac{1+s}{4a} \\ 0 & \frac{1-r}{4b} & -\frac{1+s}{4a} \end{bmatrix} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{(1+\nu)(1-2\nu)}{2(1+\nu)} \end{bmatrix}$$

$$\times \begin{bmatrix} \frac{s-1}{4a} & \frac{1-s}{4a} & \frac{1+s}{4a} & -\frac{1+s}{4a} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{r-1}{4b} & -\frac{1+r}{4b} & \frac{1+r}{4b} & \frac{1-r}{4b} \\ \frac{r-1}{4b} & -\frac{1+r}{4b} & \frac{1+r}{4b} & \frac{1-r}{4b} & \frac{s-1}{4a} & \frac{1-s}{4a} & \frac{1+s}{4a} & -\frac{1+s}{4a} \end{bmatrix} \mathrm{d}r\,\mathrm{d}s$$

المان مستطیلی چهارگرهی، کرنش صفحه‌ای

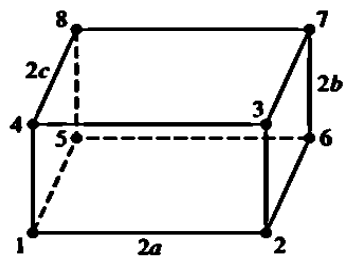


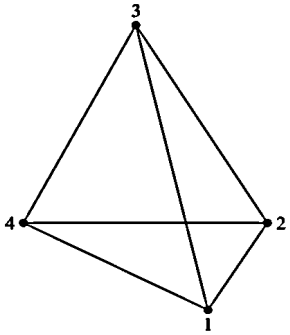
$$k_{11}^{(e)} = \frac{Etb}{16a(1+2\nu)} \int_{-1}^{1}\int_{-1}^{1} (s-1)^2 \, dr \, ds + \frac{Eta}{32b(1+\nu)} \int_{-1}^{1}\int_{-1}^{1} (r-1)^2 \, dr \, ds$$

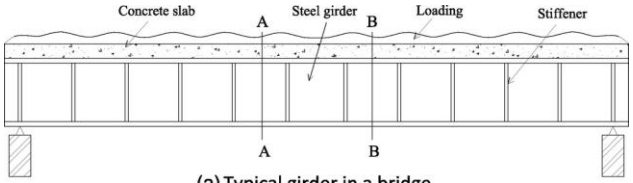
$$k_{11}^{(e)} = \frac{Etb}{16a(1+2\nu)} \frac{2(s-1)^3}{3}\bigg|_{-1}^{1} + \frac{Eta}{32b(1+\nu)} \frac{2(r-1)^3}{3}\bigg|_{-1}^{1}$$

$$= \frac{Etb}{16a(1+2\nu)}\left(\frac{16}{3}\right) + \frac{Eta}{32b(1+\nu)}\left(\frac{16}{3}\right)$$

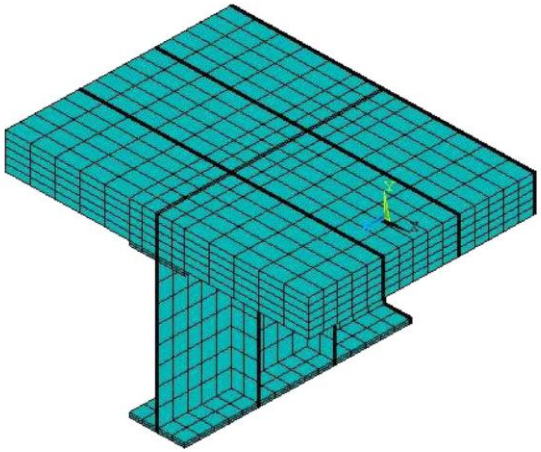
المان های تنش سه بعدی کلی

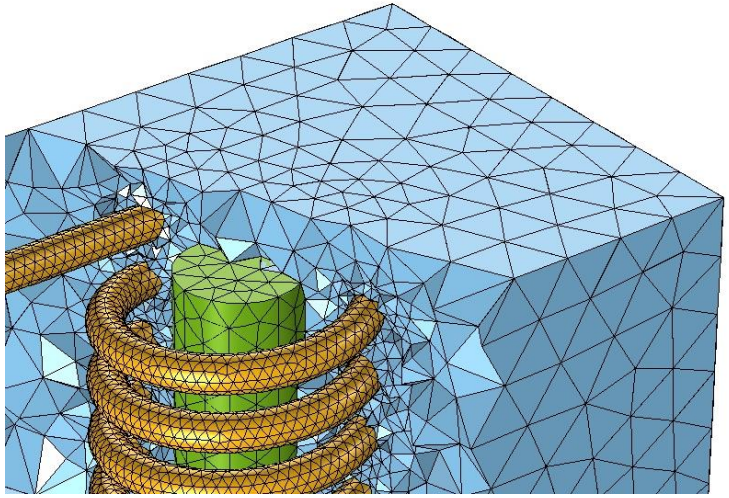






(a) Typical girder in a bridge





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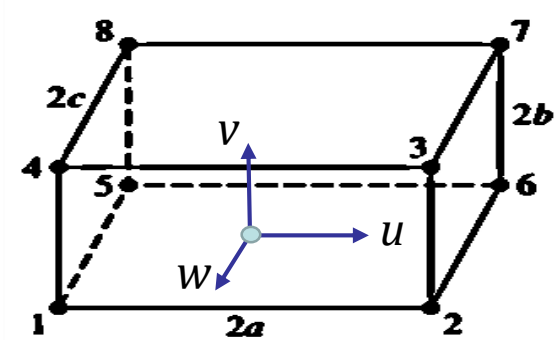
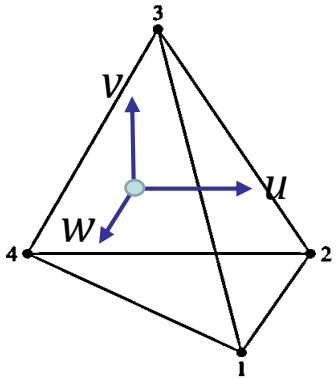
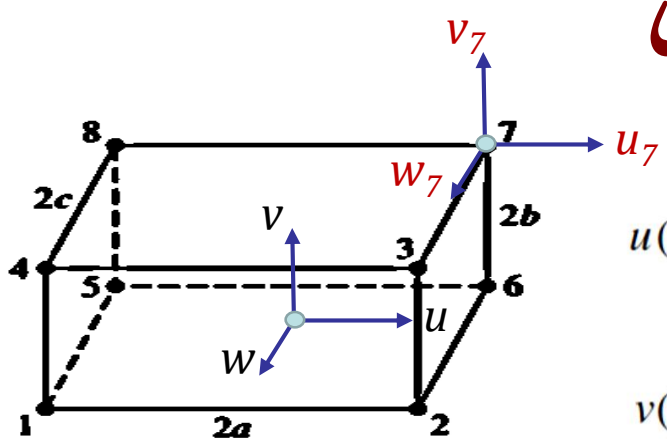
المان های تنش سه بعدی کلی





$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial w}{\partial z} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \\ \dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x} \\ \dfrac{\partial v}{\partial z} + \dfrac{\partial w}{\partial y} \end{Bmatrix} = \overbrace{\begin{bmatrix} \dfrac{\partial}{\partial x} & 0 & 0 \\ 0 & \dfrac{\partial}{\partial y} & 0 \\ 0 & 0 & \dfrac{\partial}{\partial z} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} & 0 \\ \dfrac{\partial}{\partial z} & 0 & \dfrac{\partial}{\partial x} \\ 0 & \dfrac{\partial}{\partial z} & \dfrac{\partial}{\partial y} \end{bmatrix}}^{[L]} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} \Rightarrow \{\varepsilon\} = [L] \begin{Bmatrix} u \\ v \\ w \end{Bmatrix}$$

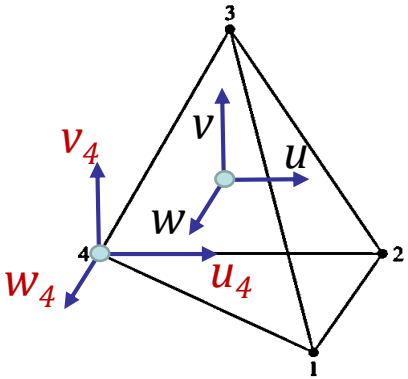
المان های تنش سه بعدی کلی



$$u(x,y,z)=\sum_{i=1}^{M} N_i(x,y,z)u_i$$

$$v(x,y,z)=\sum_{i=1}^{M} N_i(x,y,z)v_i$$

$$w(x,y,z)=\sum_{i=1}^{M} N_i(x,y,z)w_i$$



$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \overbrace{\begin{bmatrix} [N] & [0] & [0] \\ [0] & [N] & [0] \\ [0] & [0] & [N] \end{bmatrix}}^{[N_3]} \{\delta\} \Rightarrow \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = [N_3]\{\delta\} \Rightarrow \{\varepsilon\} = \overbrace{[L][N_3]}^{[B]}\{\delta\}$$

$$[N] = [N_1 \quad N_2 \quad \cdots \quad N_M]$$

$$\{\delta\} = [\,u_1 \quad u_2 \quad \cdots \quad u_M \quad v_1 \quad v_2 \quad \cdots \quad v_M \quad w_1 \quad w_2 \quad \cdots \quad w_M\,]^T$$

المان های تنش سه بعدی کلی

$$\Pi = U_e - W = \frac{1}{2}\iiint_V \{\varepsilon\}^T[D]\{\varepsilon\}\,\mathrm{d}V - \{\delta\}^T\{f\}$$

$$\{f\} = [f_{1x} \quad f_{2x} \quad \cdots \quad f_{Mx} \quad f_{1y} \quad f_{2y} \quad \cdots \quad f_{My} \quad f_{1z} \quad f_{2z} \quad \cdots \quad f_{Mz}]^T$$

$$\Pi = U_e - W = \frac{1}{2}\iiint_V \delta^T[L]^T[N_3]^T[D][L][N_3]\{\delta\}\,\mathrm{d}V - \{\delta\}^T\{f\}$$

$$\Pi = U_e - W = \frac{1}{2}\{\delta\}^T\iiint_V [B]^T[D][B]\,\mathrm{d}V\{\delta\} - \{\delta\}^T\{f\}$$

المان های تنش سه بعدی کلی

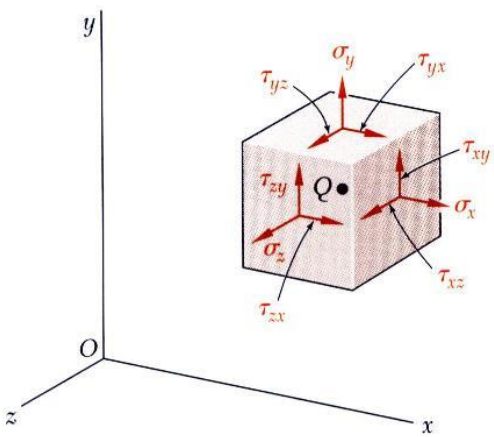
$$[B] = [L][N_3] = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial z} \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \end{bmatrix} \begin{bmatrix} [N] & [0] & [0] \\ [0] & [N] & [0] \\ [0] & [0] & [N] \end{bmatrix}$$

$$[k] = \iiint_V [B]^T [D][B]\, \mathrm{d}V$$

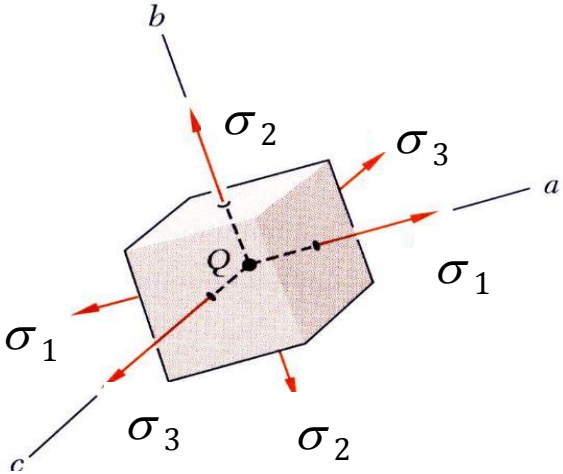
محاسبه‌ی کرنش و تنش

$$\{\varepsilon\} = [L]\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = [L][N_3]\{\delta\} = [B]\{\delta\}$$

$$\{\sigma\} = [D][B]\{\delta\}$$

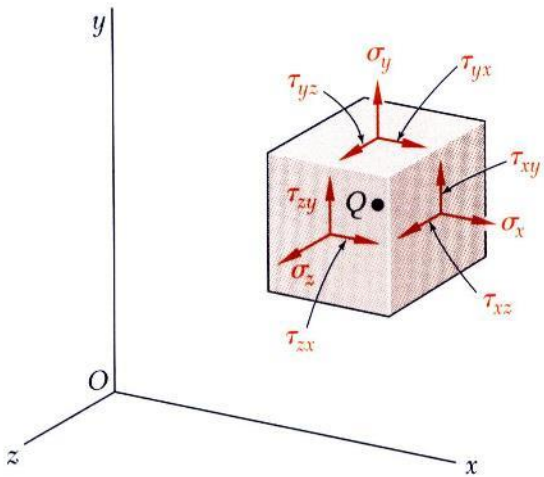


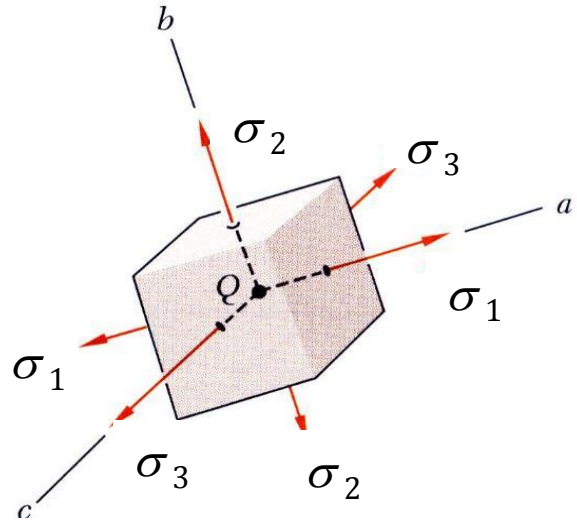
محاسبه تنش های اصلی:

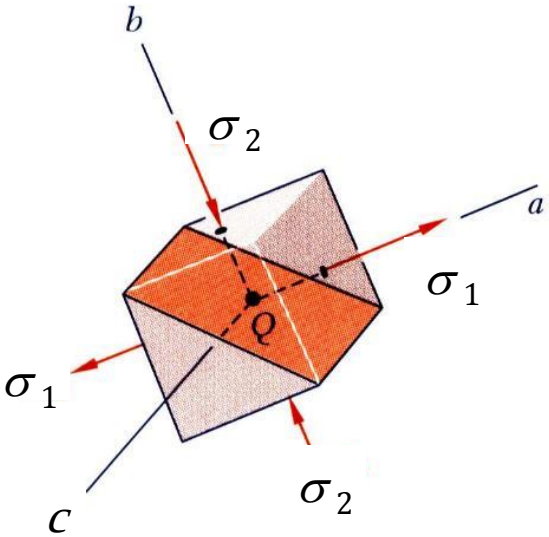


$$\begin{vmatrix} \sigma_x - \sigma & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y - \sigma & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z - \sigma \end{vmatrix} = 0$$

محاسبه‌ی کرنش و تنش





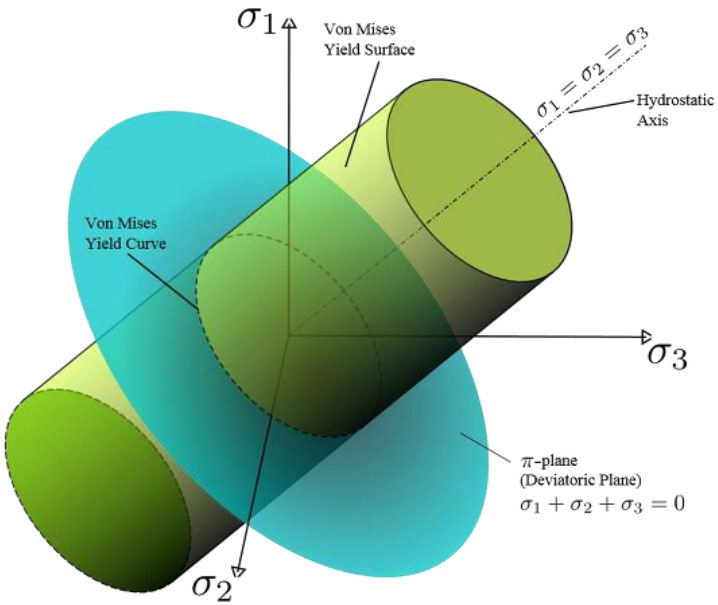


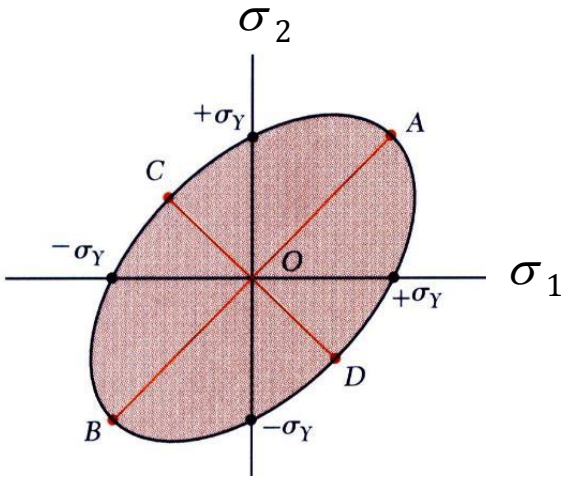
محاسبه تنش برشی حداکثر:

$$= \text{largest of}\left(\frac{|\sigma_1-\sigma_2|}{2}, \frac{|\sigma_1-\sigma_3|}{2}, \frac{|\sigma_2-\sigma_3|}{2}\right)$$

معیار خرابی

ضابطه ممانعت از شکست (تسلیم) بر اساس نظریه انرژی کرنشی اعوجاجی (یا معیار فون میسز) برای مواد شکل پذیر:





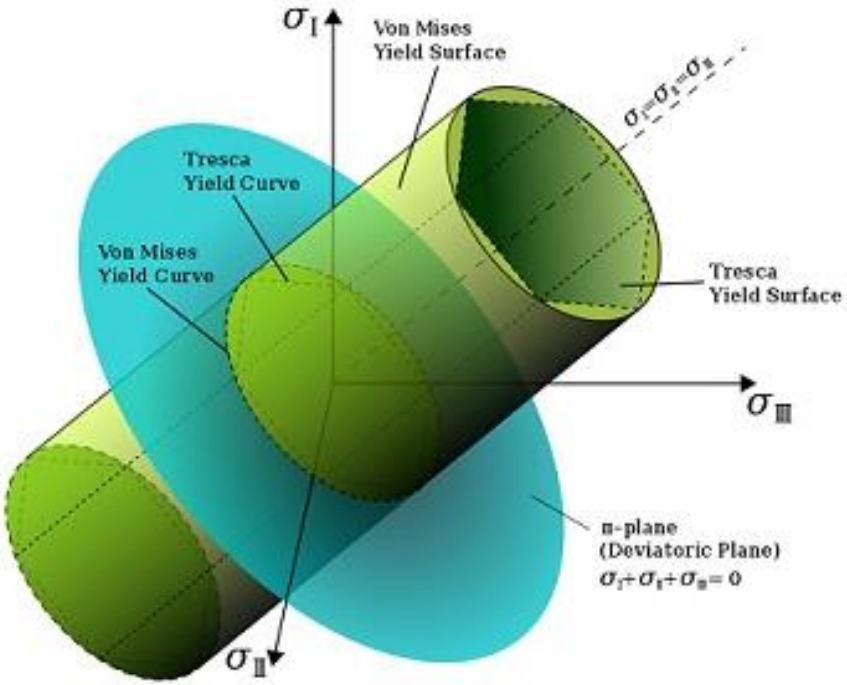
$$\sigma_e < \sigma_y$$

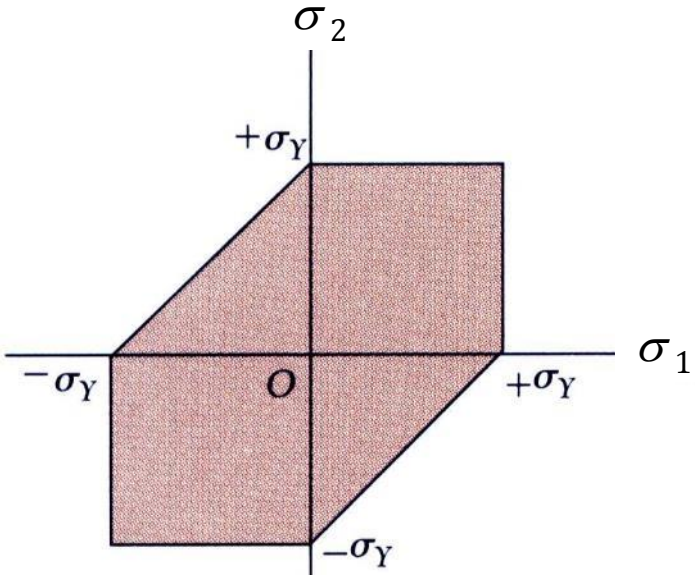
تنش فون میسز:

$$\sigma_e = \left[\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2}{2} \right]^{1/2}$$

معیار خرابی

ضابطه ممانعت از شکست (تسلیم) بر اساس نظریه تنش برشی حداکثر (یا معیار ترسکا) برای مواد شکل پذیر:





$$\tau_{max} < \sigma_y / 2$$

خلاصه

- ✓ توسعه‌ی اصلی‌ترین المان‌های محدود مورد استفاده در مکانیک جامدات در این فصل
- ✓ ارائه‌ی دید مناسبی بر روی روش‌های کلی مورد استفاده جهت توسعه‌ی معادلات المان و در نتیجه درک فرمول‌بندی مدل در حالات مختلف تنش